Proportionality in Participatory Budgeting with Type Constraints

Xingchen Sha*
Department of Computer Science
Northwestern University
xingchen.sha@u.northwestern.edu

Jeremy Vollen*
Department of Computer Science
Northwestern University
vollen@northwestern.edu

Matthew M. Casey
Department of Computer Science
Northwestern University
mattcasey@u.northwestern.edu

Edith Elkind
Department of Computer Science
Northwestern University
edith.elkind@northwestern.edu

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Abstract

We study proportional representation in approval-based participatory budgeting with type constraints, where each project may belong to multiple types, and each type is associated with a budget constraint. An important special case of our model is committee renewal, where we are given a committee of size k, and can replace up to $b \leq k$ committee members. We show that, even in this simple setting, classic approval-based voting rules may fail to output outcomes that satisfy standard proportionality axioms. To tackle this issue, we formulate type-aware proportionality axioms—τ-PJR/EJR/FJR—which define a group deserving representation in terms of the group's ability to agree on a set of projects that fit within the group's proportional share of every type budget. While these axioms are not satisfied by existing voting rules, we put forward type-aware adaptations of the Method of Equal Shares and Sequential Phragmén's rule, which we term τ-MES and τ-seqPhragmén. We prove that τ-MES satisfies τ-PJR in participatory budgeting with laminar type constraints. In multiwinner voting, τ-MES satisfies τ-EJR for nested types, whereas τ-seqPhragmén satisfies τ-PJR for laminar types. These results are essentially tight: the respective proportionality guarantees fail once the structural assumptions are relaxed. Moreover, we show that no polynomial-time rule can satisfy τ-EJR in participatory budgeting with laminar type constraints unless $P = NP$. Thus, our proposed rule achieves the strongest guarantee one can hope for in polynomial time in this setting. For arbitrary types, we obtain a powerful existence result, but it comes with computational limitations: our strongest axiom—τ-FJR—is always satisfiable, but computing a feasible outcome that satisfies even the weaker τ-PJR axiom is strongly NP-hard. We complement our theoretical analysis with experiments on real-world participatory budgeting datasets, using elections in Amsterdam with explicit category/neighborhood budgets and elections in Warsaw with synthetically generated type budgets.

1 Introduction

Participatory Budgeting (PB) has become an important mechanism for enabling citizens to participate directly in public spending decisions, and has been widely adopted by municipalities around the world, including in Brazil, the United States, France, Poland, the Netherlands, Spain, and South Africa [Cabannes, 2004, Sintomer et al., 2008, Blakey, 2008, Röcke, 2014, Masiya et al., 2021]. In a typical PB process, voters express support for projects, and the planner selects a subset of projects whose cost does not exceed a given budget. A key concern in this setting is proportionality, i.e., ensuring that groups of voters with common interests are allocated a fair share of the total budget [Aziz and Lee, 2021, Peters et al., 2021, Brill et al., 2023, Mavrov et al., 2023, Kraiczy and Elkind, 2023]; see also the surveys by Aziz and Shah [2020] and Rey et al. [2025b].

*Equal contribution.

However, in many practical PB scenarios, a single budget constraint does not fully capture the structure of feasible outcomes. Modern public agencies often allocate separate budgets across different programs, policy areas, constituencies, or target populations [Wildavsky, 1964, Rubin, 2019, Schick, 2008]. A city may also wish to limit how much can be spent on transportation, education, or public safety projects; indeed, citizens have expressed a desire for PB budgets to be fairly distributed across city districts and project categories [Yang et al., 2024]. Further, some funds may be earmarked for specific purposes and therefore may only be used to support projects of a certain type [Ebdon and Franklin, 2006]. Type-based constraints may also arise in a range of domains besides PB, such as machine learning and data mining [Cui et al., 2024]. For example, in fair clustering or data summarization, a type may represent a protected group or demographic category, with a constraint specifying how many centers or representatives are selected from that group [see, e.g., Kleindessner et al., 2019, Jones et al., 2020, Hotegni et al., 2023]. In participatory budgeting, an additional complication is that a given project may belong to multiple types—e.g., 'building a cycle path in Uptown' is both a 'transportation infrastructure project' and an 'Uptown project'—and hence may be subject to multiple type constraints. Thus, an important research challenge is to extend the mathematical framework of participatory budgeting so as to express constraints of this form, and, in particular, to formulate suitable notions of proportionality and fairness in the presence of per-type budgets.

A number of recent papers consider participatory budgeting and/or multiwinner voting (a special case of participatory budgeting, where all projects have the same cost) with constraints. Many of these works focus on the algorithmic challenges associated with maximizing the social welfare subject to constraints [Bredereck et al., 2018, Fluschnik et al., 2019, Jain et al., 2021, Hershkowitz et al., 2021, Goyal et al., 2023], extending earlier work on the multidimensional knapsack problem [Kellerer et al., 2004]. The literature on proportionality in PB with constraints is considerably more scarce, and seeks to handle a very broad class of constraints [Mavrov et al., 2023, Masařík et al., 2024]; accordingly, these works put forward general definitions of proportionality that apply well beyond settings with type-budget constraints. Our focus is complementary: by restricting attention specifically to type-based budget constraints, we can study proportionality notions tailored to this setting and rule out some outcomes that may be undesirable from the perspective of proportional access to shared type budgets, yet are permitted by these more general definitions (see Propositions B.1 and B.2 for an extended comparison). Our goal is to understand whether, by exploiting the structure of per-type budget constraints, we may be able to (1) formulate succinct and easy-to-parse axioms that are tailored to capture proportionality in this setting, and (2) propose appealing voting rules that satisfy these axioms.

Our Contribution We propose a family of proportionality axioms for *participatory budgeting with type constraints (τ-PB)*, as well as voting rules that satisfy these axioms.

In our model, projects may belong to multiple types, and each type is associated with a budget limit. The planner must choose a subset of projects so that the total spending on each type does not exceed the budget for that type. This model generalizes standard participatory budgeting (and hence multiwinner voting), while capturing a wide range of settings in which different classes of projects compete for distinct but interacting budget resources. This includes, e.g., committee selection with upper quotas, where each type represents a demographic group, region, or profession, and the budget of a type specifies the maximum number of selected candidates from that category [Bredereck et al., 2018, Celis et al., 2018].

Our starting point for defining proportionality is the family of justified representation (JR) axioms, which were first defined in the context of multiwinner voting [Aziz et al., 2017, Sánchez-Fernández et al., 2026] and then extended to participatory budgeting [Peters et al., 2021, Los et al., 2022]. At a high level, our axioms adapt the notion of justified representation to account for multiple type budgets. A group of voters is considered deserving of ℓ projects if its members commonly approve at least ℓ projects that can be funded using the group's proportional share of every type budget. As a first step, we consider the simplest variant of PB with type constraints — committee renewal — where we are given a size-k committee and are allowed to replace up to $b \leq k$ committee members. We show

(a) Existence and complexity

	Nested	Laminar	Arbitrary
Guaranteed to exist			
τ-PB	τ-FJR	τ-FJR	τ-FJR (Prop. 3)
Guaranteed in polynomial time			
τ-MW	✓ τ-EJR (Thm. 2) ✓ τ-PJR	? τ-EJR ✓ τ-PJR (Thm. 6)	✗ τ-EJR ✗ τ-PJR (Thm. 1)
τ-PB	✗ τ-EJR (Thm. 4) ✓ τ-PJR	✗ τ-EJR ✓ τ-PJR (Thm. 5)	✗ τ-EJR ✗ τ-PJR

(b) τ-MES

	Nested	Laminar	Arbitrary
τ-MW	✓ τ-EJR	✓ τ-PJR	✗ τ-PJR
τ-PB	✓ τ-PJR	✓ τ-PJR	✗ τ-PJR

(c) τ-seqPhragmén

	Nested	Laminar	Arbitrary
τ-MW	✓ τ-PJR	✓ τ-PJR	✗ τ-PJR
τ-PB	✗ τ-PJR	✗ τ-PJR	✗ τ-PJR

Table 1: Summary of our results for participatory budgeting with type constraints (τ-PB) and its special case with unit costs and integral budgets (τ-MW). Table (a) shows which axioms can always be satisfied, and which can (✓) or cannot (✗, unless $P = NP$) be guaranteed in polynomial time, with ? marking open problems. Tables (b) and (c) show the strongest axiom that τ-MES and τ-seqPhragmén satisfy, and ✗ indicates that the rule may fail τ-PJR (and hence τ-EJR). Positive results extend to special cases and weaker axioms, and negative results extend to more general settings and stronger axioms.

that, even in this simple case, classic multiwinner rules that offer strong guarantees in the 'vanilla' setting may fail to output outcomes that satisfy JR.

Motivated by these limitations, we propose type-aware analogs of common justified representation axioms, namely, τ-PJR, τ-EJR, and τ-FJR, as well as adaptations of two well-known PB voting rules with strong proportionality guarantees, namely, the Method of Equal Shares (MES) [Peters and Skowron, 2020] and Sequential Phragmén's rule [Phragmén, 1894]. We first define τ-PJR, under which an ℓ-deserving group must collectively receive at least ℓ approved projects. The more demanding τ-EJR strengthens this requirement by requiring that at least one voter in an ℓ-deserving group receives at least ℓ approved projects. Finally, τ-FJR further strengthens the guarantee by extending representation to groups whose members need not commonly approve the same set of projects, as long as there is a proportionally affordable set giving every member at least ℓ approved projects. Our adapted rules, τ-MES and τ-seqPhragmén, are polynomial-time computable and provide proportionality guarantees in a number of important settings that are more general than committee renewal. When the constraint structure forms a laminar set family, we show that τ-MES satisfies τ-PJR. For the more restrictive setting of multiwinner voting with a nested constraint structure, we prove that τ-MES satisfies the stronger τ-EJR axiom, whereas τ-seqPhragmén satisfies τ-PJR in multiwinner voting with laminar type constraints.

We also give a number of lower bounds, showing that our algorithmic results do not hold (and, in some cases, cannot be improved unless $P = NP$) under stronger proportionality properties or more general constraint structures. In particular, we show that, unless $P = NP$, there is no polynomial-time algorithm that satisfies τ-EJR in τ-PB, even for disjoint or nested constraint structures (both are special cases of laminar types). Moreover, we demonstrate that our analysis of τ-MES is tight. For any non-laminar constraint family over a given set of candidates, τ-MES may fail to satisfy τ-PJR. Similarly, we show that our τ-EJR result for multiwinner voting with nested constraints no longer holds if we generalize from multiwinner voting to heterogeneous costs, or from nested to laminar constraints.

For arbitrary types, we can no longer compute proportional outcomes in polynomial time: computing a feasible outcome that satisfies even τ-PJR (our weakest axiom) is strongly NP-hard. However, we obtain a powerful existence result by showing that τ-FJR (our strongest axiom) is always satisfiable. Table 1 summarizes these results.

We also evaluate our voting rules on real-world participatory budgeting datasets. Our experiments show that existing rules that ignore type constraints often produce infeasible outcomes. For the participatory budgeting elections in Warsaw from 2020 to 2024, our proposed rules satisfy τ-EJR in all 1,500 instances with sampled type budgets. In contrast, the greedy rule that maximizes an objective function over voter utilities violates τ-EJR in 23.1% of the runs.

2 Related Work

Proportionality in participatory budgeting The study of proportionality in participatory budgeting originates in the literature on approval-based multiwinner voting — a special case of participatory budgeting where all projects have the same cost. In that setting, the *justified representation* (JR) axiom and a number of related stronger axioms have been introduced [Aziz et al., 2017, Sánchez-Fernández et al., 2026, Peters and Skowron, 2020, Kalayci et al., 2025, Brill and Peters, 2023, Casey and Elkind, 2025]. These papers and others [Aziz et al., 2018a, Brill et al., 2024] have identified (and, in some cases, proposed) a variety of voting rules that satisfy the JR axioms, including Thiele rules, the Method of Equal Shares (MES), and Sequential Phragmén, which we use in our work. Lackner and Skowron [2023] provide a survey of recent work on proportionality in multiwinner voting.

Subsequently, this line of work was extended from multiwinner voting to participatory budgeting. To do so, one needs to specify how to compute a voter's utility from her approved project being selected: the literature distinguishes between *approval utilities* (where this utility is assumed to be 1), *cost utilities* (where this utility is assumed to be equal to the project's cost) and *general additive utilities* (where utilities may differ across voters and need to be provided for each voter-project pair). Each of these utility models has been considered in the literature. Specifically, Aziz et al. [2018b] adapted JR and PJR to the PB setting with cost utilities. Peters et al. [2021] studied general additive utilities in PB: they adapted EJR to this setting, introduced FJR, and proved that MES satisfies EJR for approval utilities. Subsequently, Los et al. [2022] extended the definition of PJR to general additive utilities and proved that Phragmén satisfies PJR. Brill et al. [2023] studied a restricted family of general additive utility functions, showing positive and negative results for when the JR axioms can be satisfied for all of these utility functions simultaneously. For surveys on participatory budgeting, see Rey et al. [2025b] and Aziz and Shah [2020].

Participatory budgeting with constraints Several papers have studied participatory budgeting with constraints similar to ours. Jain et al. [2021] consider PB with project groups, where projects are partitioned into groups and each group has its own budget limit, and study the computational complexity of maximizing welfare. Hershkowitz et al. [2021] study the problem of maximizing social welfare in participatory budgeting under the constraint that each district in the city is entitled to at least some portion of the budget. Chen et al. [2022] study PB with donations, where voters can donate their own money to fund projects, and various axioms related to these donations. In their model they also allow project types and lower/upper quotas on the types. Rey et al. [2025a] propose a general framework for PB with constraints, and embed the problem into judgment aggregation. They focus on satisfying a set of monotonicity axioms. Goyal et al. [2023] also study a model of PB with project types, but focus on strategic and welfare considerations. Fairstein et al. [2021] study a model of PB with substitute projects, where voters can express which projects are substitutes. They also give an adaptation of MES, and show it satisfies an up-to-one relaxation of an adapted version of EJR. However, in their model each project has a single type, whereas our framework allows projects to belong to multiple types. Hence, both their model and axioms do not capture our setting with type constraints.

Proportionality under feasibility constraints Most closely related to our work are a series of papers studying proportionality axioms in participatory budgeting under general feasibility constraints [Mavrov et al., 2023, Masařík et al., 2024, Skowron, 2026]. In these papers, the PB instance comes with a set of feasible sets $\mathcal{F}$, which denote which outcomes are allowable. This general approach captures our problem, if we take $\mathcal{F}$ to be the set of outcomes that respect all type budgets. However, their model is also more general, allowing for more complicated feasibility constraints, such as lower quotas requiring that we have a certain number of projects from a specific type. Mavrov et al. [2023] introduced the restrained-EJR axiom as a generalization of EJR to this setting. Masařík et al. [2024] then introduced an alternative definition of EJR for this setting, proving that it implies restrained-EJR. Skowron [2026] then generalized this setting to one with additive utilities and generalized the latter definition of EJR.

The axioms from these papers adopt a different perspective on fairness/proportionality than ours. Our axioms put an emphasis on *equality of voting power*, while their axioms emphasize *equality of outcome*. The two approaches are therefore generally incomparable: the general feasibility-based axioms can provide stronger guarantees on some instances, while on others they permit undesirable outcomes that our notions rule out. We demonstrate this incomparability in Propositions B.1 to B.3.

Their approach does not consider any normative meaning of the type constraints, treating them simply as constraints on the way utility can be distributed among voters. Accordingly, under their definition, given a situation where it is only feasible to include one of two groups of candidates T_1, T_2 in the outcome, the group which has a larger supporter:group size ratio will be included. In particular, this implies that for two groups of voters of the same size, each approving their own group of candidates, their axioms aim to equalize the representation received by the two voter groups. They encode fairness with respect to the distribution of welfare—each group's utility in the outcome should proportionally reflect their size.

Our axioms take a different view, focusing on the constraints as normatively important in their own right. Our axioms are predicated on the view that each resource type constraint is a separate decision to be decided upon by the voters. Accordingly, we require that a voter group's power to make decisions about any individual type be proportional to their size. We encode fairness as equally distributing the voting power *of each type* among the voters.

The distinction between fairness notions focusing on distributing voter welfare and voter power was made in Peters and Skowron [2020].[1] They classify PAV as aiming for fairness of welfare distribution, and MES/Sequential Phragmén as aiming for fairness of voter power distribution. Our findings reinforce this distinction. PAV satisfies the welfarist fairness notion from the above papers, and fails our voter power based notions. Additionally, MES and Sequential Phragmén are easily adapted to reflect our splitting of voter power on a per-type basis. We believe our approach to fairness is more appropriate in settings where voters care not only about total number of candidates, but also which candidates win in each type. In particular, if approved candidates of different types are not substitutes for one another, our axioms may actually reflect the underlying utilities of voters more appropriately. Conversely, for settings with types that only encode feasibility and are not task-relevant, e.g. when approved candidates of different types are substitutes, then the welfarist fairness notions may be more appropriate.

Separately, Motamed et al. [2022] study PB where projects have costs for multiple resources and each resource has a budget—a general model that can capture our problem. They introduce a proportionality axiom that uses a similar idea to ours, requiring voters to proportionally deserve enough budget in each resource. However, their axiom is considerably weaker than ours as it is only binding for groups of voters with identical preferences, whereas our axioms also guarantee representation to arbitrary voter groups that approve candidates in common.

Related Problems A special case of our model is that of dynamic multiwinner voting, where the goal is to update a previously selected or partially selected committee while limiting the changes made to the starting committee [Zech et al., 2024, Dong et al., 2026, Dong and Peters, 2025, Elkind et al.,

[1] We thank Piotr Skowron for identifying this connection to our work.

2024, Cheng et al., 2020]. Another special case of our model is temporal voting, where there is a sequence of elections each electing one candidate and the aim is to choose a sequence of winners that satisfies JR-style axioms [Bulteau et al., 2021, Chandak et al., 2026, Elkind et al., 2025, Phillips et al., 2026, Teh, 2026]. The problem of maximizing voters' utilities in our model is closely related to multi-dimensional knapsack problems [Kellerer et al., 2004]. A number of papers have studied diversity and feasibility constraints in multiwinner voting with ordinal preferences [Bredereck et al., 2018, Celis et al., 2018]. Chingoma et al. [2024] study public decisions, a special case of PB with general feasibility constraints, introducing JR axioms that are tailored to their setting.

3 Model

In this section, we formally define the model, review existing proportionality axioms and their limitations in a simple special case of our setting, and then introduce our new proportionality axioms.

3.1 Participatory Budgeting with Type Constraints (τ-PB)

We consider approval-based participatory budgeting with type constraints. For each $s \in \mathbb{N}$, let $[s] = \{1, 2, \ldots, s\}$. An *instance* is a tuple $\mathcal{I} = (N, C, \mathrm{cost}, (A_i)_{i \in N}, d, \mathcal{H}, (b_t)_{t \in [d]})$, where:

- $N = [n]$ is a set of n *voters*;
- $C = \{c_1, \ldots, c_m\}$ is a set of m *candidates*, or *projects*;
- $\mathrm{cost} : C \to \mathbb{Q}_{>0}$ is a function that assigns to each candidate $c \in C$ its cost. For each $C' \subseteq C$, we write $\mathrm{cost}(C') = \sum_{c \in C'} \mathrm{cost}(c)$ for the total cost of C';
- for each voter $i \in N$, the set $A_i \subseteq C$ is the *approval set* of voter i;
- $d \in \mathbb{N}$ is the number of *candidate types*;
- $\mathcal{H} = \{T_1, \ldots, T_d\}$ is a *constraint structure*, which associates each $t \in [d]$ with a set of candidates $T_t \subseteq C$; we refer to elements of $\mathcal{H}$ as *candidate types*, or simply *types*;
- for each $t \in [d]$, the quantity $b_t \in \mathbb{Q}_{>0}$ is the *type budget* for type T_t.

We refer to the special case of our model where $\mathrm{cost}(c) = 1$ for all $c \in C$ and $b_t \in \mathbb{N}$ for all $t \in [d]$ as *multiwinner voting with type constraints* (τ-MW).

We refer to subsets of C as *outcomes*, or *committees*. Given an outcome $W \subseteq C$ and a type $T_t \in \mathcal{H}$, let $W_t = W \cap T_t$ denote the set of selected candidates of type T_t. We say that W is *feasible* if $\mathrm{cost}(W_t) \leq b_t$ for every $t \in [d]$.

We assume that voters have approval utilities, i.e., each voter values an outcome according to the number of selected candidates that the voter approves: formally, the utility of a voter $i \in N$ from an outcome $W \subseteq C$ is $u_i(W) = |A_i \cap W|$. A *voting rule*, or simply a *rule*, is a function that maps each instance to a non-empty set of feasible outcomes.

To illustrate the expressive power of our model, we show that it can capture three important settings.

Participatory Budgeting and Multiwinner Voting. The classic model of approval-based participatory budgeting corresponds to $d = 1$, $T_1 = C$. Then an outcome $W \subseteq C$ is feasible if and only if $\mathrm{cost}(W) \leq b_1$. If furthermore every candidate has unit cost and $b_1 \in \mathbb{N}$, our model reduces to multiwinner voting. Thus, to represent an instance of multiwinner voting, we omit costs, constraints and budgets from the notation, and write $\mathcal{I} = (N, C, (A_i)_{i \in N}, k)$, where by convention $k \in \mathbb{N}$ is the budget constraint for type $T_1 = C$, i.e., an outcome W is feasible as long as it satisfies $|W| \leq k$.

Multiwinner Voting with Upper Quotas. In approval-based multiwinner voting with upper quotas, the goal is to select a committee of size at most k, while respecting upper bounds on the number of selected candidates from different categories, such as demographic groups, regions, or professions [Bredereck et al., 2018, Celis et al., 2018, Lang and Skowron, 2018, Bei et al., 2022]. Our model can represent this setting as follows. We assign unit cost to each candidate, and create a global type $T_0 = C$ with budget $b_0 = k$. For each category g, we create a type $T_g \subseteq C$ containing the candidates in that category, with budget b_g equal to the corresponding upper quota. Then a feasible outcome is exactly a committee of size at most k that satisfies all upper quotas.

Committee Renewal/Reconfiguration. In committee renewal, the goal is to update a previously selected committee so as to reflect the voters' evolving preferences while limiting the number of replaced candidates [Cheng et al., 2020, Bredereck et al., 2022, Elkind et al., 2024, Zech et al., 2024, Dong and Peters, 2025, Dong et al., 2026]. In this case, we are given an initial committee $W_0 \subseteq C$ of size k. We are then allowed to replace up to b candidates from W_0 for some $b \leq k$. To capture this setting within our model, we define two candidate types, namely, $T_1 = C$, $T_2 = C \setminus W_0$, and $b_1 = k$, $b_2 = b$. Then an outcome $W \subseteq C$ is feasible if and only if $|W| \leq k$ and $|W \setminus W_0| \leq b$. Since this setting uses only one type constraint in addition to the global budget constraint, we henceforth refer to it as *multiwinner voting with a single type constraint (*MW-Single*).* Note that when $b = k$, MW-Single reduces to multiwinner voting.

In what follows, we discuss existing axioms (and their natural variants) for PB, and show that negative results already arise in the special case of MW-Single. For simplicity, we represent an instance of MW-Single by the tuple $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$.

3.2 Standard Proportionality Axioms and Limitations

Most prior work on proportional representation in multiwinner voting and PB with approval ballots is based on the hierarchy of *justified representation* axioms [Aziz et al., 2017, Sánchez-Fernández et al., 2026, Peters et al., 2021]. We define two of these axioms here to illustrate how they interact with constraints; in what follows, we extend these axioms (as well as two additional axioms) to the typed setting. Throughout this paper, we assume voter groups are nonempty.

Definition 1 (JR and EJR)**.** Given an instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, k)$ of multiwinner voting and an $\ell \in [k]$, we say that a group of voters $S \subseteq N$ is *ℓ-cohesive* if $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\bigcap_{i \in S} A_i\right| \geq \ell$. For a given $\ell \in [k]$, we say that a feasible committee W satisfies *ℓ-justified representation* (ℓ-JR) if, for every ℓ-cohesive group of voters $S \subseteq N$, there exists a voter $i \in S$ with $u_i(W) \geq \ell$. A feasible committee satisfies *justified representation* (JR) if it satisfies 1-JR; it satisfies *extended justified representation* (EJR) if it satisfies ℓ-JR for all $\ell \in [k]$. A rule $\mathcal{R}$ *satisfies* JR/EJR if, on all instances, every outcome it returns satisfies JR/EJR.

These definitions apply to τ-MW as long as the constraint structure $\mathcal{H}$ includes a type T_t with $T_t = C$, $b_t = k$: we simply need to change the interpretation of 'feasible' from '$|W| \leq k$' to '$\text{cost}(W_t) \leq b_t$ for every $T_t \in \mathcal{H}$'.

In the multiwinner setting without types, committees satisfying JR or EJR always exist and can be computed in polynomial time [Aziz et al., 2018a, Peters and Skowron, 2020]. In contrast, there are instances of τ-MW where no feasible committee satisfies JR. Indeed, this is already true for MW-Single: let $n = k = 2$, $A_1 = \{x\}$, $A_2 = \{y\}$, and set $C = \{u, w, x, y\}$, $W_0 = \{u, w\}$, $b = 1$. Then the only size-2 subset of C that satisfies JR is $\{x, y\}$, but it is not feasible since $b = 1$.

One can then aim to output feasible JR committees whenever they exist. However, in Appendix A, we show that the associated computational problems (such as, e.g., deciding if an instance of τ-MW admits a JR committee) are computationally hard even for MW-Single. Moreover, a natural approach to finding JR/EJR committees may fail in the presence of constraints. Specifically, Thiele rules (defined below) may fail to select JR committees when applied to τ-MW elections, even if these elections admit a feasible outcome that satisfies JR; this holds even for the PAV rule, which satisfies EJR in the standard model.

Definition 2 (Thiele rules)**.** A *weight function* is a mapping $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(1) = 1$ and $f(s) \geq f(s+1)$ for every $s \in \mathbb{N}$. Given a feasible committee W and a weight function f, we define the *f-Thiele score* of W as $\text{score}_f(W) = \sum_{i \in N} \sum_{s=1}^{|A_i \cap W|} f(s)$. The f-Thiele rule outputs feasible committees that maximize the f-Thiele score. One of the most prominent Thiele rules is Proportional Approval Voting (PAV), which is given by $f(s) = \frac{1}{s}$ for all $s \in \mathbb{N}$.

In the standard multiwinner setting, all Thiele rules with $f(s) \leq \frac{1}{s}$ for all $s \in \mathbb{N}$ satisfy JR, and PAV satisfies EJR [Aziz et al., 2017]. One may then expect that when such rules are forced to choose from feasible outcomes of τ-MW, and some such outcome satisfies JR, the rule will be able to find a JR-satisfying outcome. We will now show that this is not the case, even for MW-Single.

Proposition 1. *There exists an instance of* MW-Single *that admits a JR committee for which no Thiele rule returns a committee satisfying JR.*

Proof. Consider the instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ defined as follows. Let $C = \{c_i : 1 \leq i \leq 7\}$, set $k = 4$, let $W_0 = \{c_1, c_2, c_3, c_4\}$, and set the replacement budget to $b = 2$. We create $n = 24$ voters (so $n/k = 6$). One voter approves $\{c_1, c_5\}$, five voters approve $\{c_5\}$, seven voters approve $\{c_6\}$, seven voters approve $\{c_7\}$, two voters approve $\{c_3\}$, and two voters approve $\{c_4\}$. Fix any Thiele weight function f; note that $f(1) = 1$.

Consider the committee $W^\star = \{c_3, c_4, c_6, c_7\}$. Since $W^\star \setminus W_0 = \{c_6, c_7\}$ and $b = 2$, this committee is feasible. Its Thiele score is 18, as there are 18 voters who each approve a single member of $W^\star$.

Now, let W be a feasible committee that satisfies JR. We will argue that its Thiele score is at most 17. Note first that W must contain c_6 and c_7, since the seven voters approving c_6 form a 1-cohesive group, and so do the seven voters approving c_7. Further W intersects $\{c_1, c_5\}$, since there are six voters who all approve c_5. Since $b = 2$ and $c_5, c_6, c_7 \notin W_0$, it cannot be the case that $c_5 \in W$. Hence, $c_1 \in W$. With c_1, c_6, c_7 fixed, only one slot remains. This slot can represent at most two additional voters, by choosing either c_3 or c_4. Therefore, every feasible JR committee represents at most $1 + 7 + 7 + 2 = 17$ voters, and hence has Thiele score at most 17.

Thus, under every Thiele rule $W^\star$ has a strictly higher score than any feasible JR committee. □

3.3 Type-Aware Proportionality Axioms

We have seen that there are instances of MW-Single that admit no JR committees. The key reason for this is that in the standard PB model, a group of voters may claim representation whenever it can identify a set of commonly approved candidates X that it can afford using its proportional share of the *overall* budget; crucially, X need not be affordable with respect to the group's share of type budgets. However, intuitively, in the presence of constraints, a group should be regarded as deserving representation only if it can propose a set of candidates that fits within the group's proportional share of *every* type budget. We will now formalize this idea of 'deservingness' and the resulting notions of justified representation.

Definition 3. Given an instance $\mathcal{I}$ of τ-PB and an $\ell \in \mathbb{N}$, we say that a group of voters $S \subseteq N$ is *ℓ-deserving* if there exists a set $X \subseteq \bigcap_{i \in S} A_i$ such that $|X| \geq \ell$ and, for every type $T_t \in \mathcal{H}$, we have $\text{cost}(X \cap T_t) \leq b_t \cdot \frac{|S|}{n}$.

Proposition 2. *In the special case of multiwinner voting, a group of voters is ℓ-deserving if and only if it is an ℓ-cohesive group in the sense of Definition 1.*

We can now use this notion of an ℓ-deserving group to generalize EJR to τ-PB. We note that the JR axiom, which we defined in the multiwinner setting, does not have a natural analog in the PB setting. We will also introduce an adaptation of the PJR axiom [Sánchez-Fernández et al., 2026, Los et al., 2022], which is slightly weaker than EJR (but stronger than JR in the multiwinner setting).

Definition 4 (Proportional/Extended Justified Representation with Type Constraints (τ-PJR/EJR))**.** Given a τ-PB instance $\mathcal{I}$, a feasible outcome W satisfies τ-PJR (resp., τ-EJR) if for every integer

$\ell \geq 1$ and every ℓ-deserving group of voters $S \subseteq N$ it holds that $\left|W \cap \bigcup_{i \in S} A_i\right| \geq \ell$ (resp., there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$). A rule $\mathcal{R}$ satisfies τ-PJR (resp., τ-EJR) if all the outcomes returned by R satisfy τ-PJR (resp., τ-EJR).

In the PB setting without candidate types, EJR can be strengthened to *full justified representation (FJR)*, by weakening the formulation of cohesiveness [Peters et al., 2021]. In the same spirit, we can weaken the notion of an ℓ-deserving group (so that there are more groups of voters that are entitled to representation) and define a type-aware version of FJR (τ-FJR).

Definition 5. Given an instance $\mathcal{I}$ of τ-PB and an $\ell \in \mathbb{N}$, we say that a group of voters $S \subseteq N$ is *weakly ℓ-deserving* if there exists a set $X \subseteq C$ such that, for every voter $i \in S$, we have $|A_i \cap X| \geq \ell$, and for every type $T_t \in \mathcal{H}$ we have $\text{cost}(X_t) \leq b_t \cdot \frac{|S|}{n}$.

In other words, a group weakly deserves ℓ candidates if there exists a set of candidates that gives every voter in the group utility at least ℓ, while staying within the group's proportional share of every type budget. We can now define τ-FJR, which guarantees representation to all weakly deserving groups, and thus strengthens τ-EJR.

Definition 6 (Full Justified Representation with Type Constraints (τ-FJR))**.** A feasible outcome W satisfies τ-FJR if for every integer $\ell \geq 1$ and every group of voters $S \subseteq N$ that weakly deserves ℓ candidates in τ-*PB*, there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$.

In the multiwinner voting setting, every instance admits a committee that satisfies FJR. In what follows, we will show that this result extends to our setting.

3.4 Comparison to Prior Work

In Appendix B, we discuss three related extensions of EJR to settings with general feasibility constraints, namely, Restrained EJR (proposed by Mavrov et al. [2023]) and Base EJR and EJR with Feasibility Constraints (put forward by Masařík et al. [2024]). These notions reflect a perspective on proportionality that is different from ours. Our axioms determine a group's entitlement from its proportional share of each shared type budget, independently of the selected outcome, whereas the axioms of Mavrov et al. [2023] and Masařík et al. [2024] define entitlement in terms of feasible augmentations of partial committees. Consequently, neither family of axioms dominates the other, even in the special case of committee renewal; in Appendix B, we provide concrete examples that illustrate this. This difference between the axioms is also reflected in the voting rules that satisfy them. Specifically, the PAV rule is known to satisfy the EJR axioms of Mavrov et al. [2023] and Masařík et al. [2024]. In contrast, we will show that no Thiele rule, including PAV, guarantees τ-PJR (and hence τ-EJR) even in the MW-Single setting (Proposition 8).

4 General Model: Existence and Complexity

Our type-aware axioms may appear to be quite demanding. It is therefore not obvious if they are always satisfiable. We will now show that this is indeed the case, even for the strongest of these axioms, namely, τ-FJR.

In the standard PB setting (i.e., a single budget constraint), FJR is known to be satisfied by the Greedy Cohesive Rule (GCR) [Peters et al., 2021]. It is straightforward to extend GCR to our setting; we refer to the resulting rule as τ-GCR. It starts with an empty outcome $W = \emptyset$ and labels all voters as active. Iteratively, it looks for the maximum value of ℓ such that there exists a group S of active voters that weakly deserves ℓ candidates (as witnessed by a candidate set $X(S) \subseteq C$). Among all weakly ℓ-deserving groups S, it picks one with the smallest value of $\text{cost}(X(S))$, adds $X(S)$ to W, and labels all voters in S as inactive. It terminates when there are no weakly 1-deserving groups of active voters. We will now argue that this procedure guarantees τ-FJR.

Proposition 3. *In τ-PB with arbitrary type constraints, all outputs of τ-GCR are feasible and satisfy τ-FJR.*

Proof. First, we note that τ-GCR always returns a feasible outcome. Indeed, in each iteration j, τ-GCR deactivates a group of voters S_j and adds a set of candidates $X(S_j)$ to W. By the choice of $X(S_j)$, for every $t \in [d]$ it holds that the set $X_t(S_j) = X(S_j) \cap T_t$ satisfies $\mathrm{cost}(X_t(S_j)) \leq b_t \cdot |S_j|/n$. Suppose that τ-GCR terminates after q iterations, so that $W = \bigcup_{j\in[q]} X(S_j)$; then for each $t \in [d]$ we have $W_t = W \cap T_t = \bigcup_{j\in[q]} X_t(S_j)$. Note that the sets S_j, $j \in [q]$, are pairwise disjoint and hence $\sum_{j\in[q]} |S_j| \leq n$. Therefore for each $t \in [d]$ we have

$$\mathrm{cost}(W_t) \leq \sum_{j\in[q]} \mathrm{cost}(X_t(S_j)) \leq \sum_{j\in[q]} b_t \cdot \frac{|S_j|}{n} = \frac{b_t}{n} \sum_{j\in[q]} |S_j| \leq \frac{b_t}{n} \cdot n = b_t,$$

i.e., W is feasible.

Now, to see that τ-GCR satisfies τ-FJR, consider a weakly ℓ-deserving group of voters $S \subseteq N$, as certified by a set of candidates X. Let $i \in S$ be the first voter in S who is marked inactive during the execution of τ-GCR. Before i is marked inactive, every voter in S is active. Since S is weakly ℓ-deserving, the group S' inactivated during that iteration (which contains i) must be weakly ℓ'-deserving for $\ell' \geq \ell$. Thus $|A_i \cap W| \geq \ell' \geq \ell$, and hence W satisfies τ-FJR. □

Note that τ-GCR runs in exponential time. For the multiwinner voting setting, it has recently been established that FJR admits a polynomial-time rule [Frank and Peters, 2026]. However, we cannot expect to extend this rule to our setting: we will now show that in the presence of arbitrary type constraints, no polynomial time rule can guarantee even τ-PJR unless $P = NP$; this hardness result holds even in the multiwinner voting setting with a single voter.

Theorem 1. *Unless* P = NP*, no polynomial-time rule satisfies τ-PJR in τ-PB with arbitrary type constraints, even in the special case of τ-MW and $n = 1$.*

Proof. We reduce from Independent Set. An instance of this problem is given by an undirected graph $G = (V, E)$ and an integer κ; it is a yes-instance if G has an independent set of size at least κ and a no-instance otherwise. Given an instance (G, κ) of Independent Set, we construct a single-voter, τ-MW instance $\mathcal{I} = (N, C, \mathrm{cost}, (A_i)_{i\in N}, d, \mathcal{H}, (b_t)_{t\in[d]})$ as follows.

Let $N = \{1\}$, $C = \{c_v : v \in V\}$. All candidates have unit cost, and the unique voter approves all candidates, so $A_1 = C$. Let $d = |E|$ and write $E = \{e_1, \ldots, e_d\}$. For each $t \in [d]$ with $e_t = \{u, v\}$, create a type $T_t = \{c_u, c_v\}$ with budget $b_t = 1$.

Note that a set $W \subseteq C$ is feasible if and only if for every edge $e = \{u, v\}$, at most one of c_u, c_v belongs to W. Thus feasible outcomes correspond exactly to independent sets in G.

We claim that G admits an independent set of size at least κ if and only if all feasible outcomes that satisfy τ-PJR are of size at least κ. Indeed, if our instance admits a feasible outcome $W = \{c_v : v \in V'\}$ with $|W| \geq \kappa$ then V' is an independent set of size at least κ. Conversely, if G has an independent set V' with $|V'| \geq \kappa$, then the set $X = \{c_v : v \in V'\}$ witnesses that $S = N = \{1\}$ is $|V'|$-deserving, since $|X| = |V'|$ and the unique voter in S is entitled to the full budget of every type. Thus, if an outcome W satisfies τ-PJR, it must contain at least $|V'| \geq \kappa$ candidates approved by voter 1, i.e., be of size at least $|V'| \geq \kappa$.

Accordingly, a polynomial-time voting rule that satisfies τ-PJR can be transformed into a polynomial-time algorithm for Independent Set: the graph G admits an independent set of size at least κ if and only if on the instance $\mathcal{I}$ constructed above this rule returns a committee of size at least κ. □

Theorem 1 establishes that one cannot expect positive algorithmic results for arbitrary type constraints. It is therefore natural to ask if one can compute proportional outcomes in polynomial time under additional assumptions on $\mathcal{H}$. Note that in the construction used for the proof of Theorem 1, each type comprises exactly two candidates, so we cannot expect to obtain positive results by constraining the type sizes. Instead, we pursue a different approach: namely, we impose structural constraints on $\mathcal{H}$.

Specifically, we start by defining type-aware versions of two polynomial-time voting rules with good proportionality guarantees in the standard model, namely, MES and seqPhragmén (see Section 5).

Algorithm 1 Method of Equal Shares for Type Constraints (τ-MES)

$W \leftarrow \emptyset$
for all $i \in N$ and all $t \in [d]$: $p_i^{(t)} \leftarrow \frac{b_t}{n}$
while true **do**
  for each $c \in C \setminus W$ and each $t \in [d]$ **do**
    If $c \in T_t$, set $\Psi_t(c) = \left\{\eta \geq 0 : \sum_{i \in N : c \in A_i} \min\{\eta, p_i^{(t)}\} \geq \text{cost}(c)\right\}$
    Define $\eta_t(c)$ as

$$\eta_t(c) = \begin{cases} 0, & \text{if } c \notin T_t, \\ +\infty, & \text{if } c \in T_t \text{ and } \Psi_t(c) = \emptyset, \\ \min \Psi_t(c), & \text{if } c \in T_t, \Psi_t(c) \neq \emptyset \end{cases}$$

  end for
  Let $C^+ = \{c \in C \setminus W : \Psi_t(c) \neq \emptyset \text{ for every } t \in [d] \text{ such that } c \in T_t\}$
  If $C^+ = \emptyset$, **break**
  Pick a candidate $c^\star \in \arg\min_{c \in C^+} \max_{t \in [d]} \eta_t(c)$, breaking ties arbitrarily
  $W \leftarrow W \cup \{c^\star\}$
  for each $t \in [d]$ with $c^\star \in T_t$ **do**
    Update $p_i^{(t)} \leftarrow p_i^{(t)} - \min\{\eta_t(c^\star), p_i^{(t)}\}$ for all $i \in N$ s.t. $c^\star \in A_i$
  end for
end while
return W

We then consider two families of constraint structures: nested types (Section 6) and laminar types (Section 7), and explore whether the type-aware versions of MES and seqPhragmén (as well as Thiele rules, defined earlier in the paper) output proportional outcomes on instances with nested or laminar types. For nested types, the answer is largely positive, but for the more general class of laminar types the picture is more nuanced.

5 Type-Aware Voting Rules

We will now present adaptations of two common polynomial-time computable voting rules with strong proportionality properties, namely, MES and seqPhragmén, to the setting with types.

5.1 Method of Equal Shares with Type Constraints (τ-MES)

The Method of Equal Shares (MES) has recently emerged as one of the most widely studied rules in approval-based participatory budgeting and multiwinner voting. It has polynomial running time and is known to satisfy EJR in the standard PB setting [Peters and Skowron, 2020, Peters et al., 2021]. We will now introduce τ-MES — an extension of MES to the setting of τ-PB.

At a high level, τ-MES allocates each voter an equal share of every type budget. A candidate is affordable if and only if its approving voters can collectively cover its cost within every type to which it belongs. The rule then selects an affordable candidate so as to minimize the maximum payment required from its supporters, over all types, subject to payments for a candidate being as equal as possible across its supporters. For details, see Algorithm 1.

We will analyze the proportionality guarantees of τ-MES in subsequent sections; here, we establish that it returns a feasible outcome.

Proposition 4. *Given an instance of τ-PB, τ-MES always returns a feasible outcome.*

Proof. Consider an outcome W output by τ-MES on an instance $\mathcal{I}$ of τ-PB. To prove that the outcome is feasible, it suffices to show that for every type $T_t \in \mathcal{H}$, the budget constraint for T_t is

met: $\text{cost}(W_t) \leq b_t$. When τ-MES adds a candidate to W, it splits the cost of the candidate evenly across its approvers subject to not overspending their remaining budgets. Thus, we know that for any type T_t, the spending on T_t is bounded by the combined type-t budgets of all voters. Therefore, $\text{cost}(W_t) \leq n \cdot \frac{b_t}{n} = b_t$. □

5.2 Sequential Phragmén's Rule with Type Constraints (τ-seqPhragmén)

Another well-known and widely studied polynomial-time computable rule in the PB literature is Sequential Phragmén's rule (seqPhragmén) [Phragmén, 1894], which has been shown to satisfy PJR, but not EJR, in standard PB [Los et al., 2022]. We will now define a variant of seqPhragmén that can handle type constraints; we will refer to this rule as τ-seqPhragmén.

For each type $T_t \in \mathcal{H}$, each voter earns type-T_t currency continuously at rate $\frac{b_t}{n}$. Initially, all balances are 0, and the rule constructs an outcome W, starting from $W = \emptyset$. For each voter $i \in N$ and each $t \in [d]$, let $x_i^t(\theta)$ denote the amount of type-T_t currency held by voter i at time θ. For each candidate $c \in C$, let $\tau(c) = \{T \in \mathcal{H} : c \in T\}$ be the set of types containing c, and let $N(c) = \{i \in N : c \in A_i\}$.

An unselected candidate $c \in C \setminus W$ is *affordable* at time θ if $\sum_{i \in N(c)} x_i^t(\theta) \geq \text{cost}(c)$ for every type $T_t \in \tau(c)$; c is *eligible* with respect to the current outcome W if $N(c) \neq \emptyset$ and $\text{cost}(W_t) + \text{cost}(c) \leq b_t$ for every type $T_t \in \tau(c)$. At the earliest time when some unselected candidate is both affordable and eligible, the rule selects one such candidate, breaking ties according to a fixed tie-breaking order.

After selecting c, the voters in $N(c)$ pay for c separately in each type $T_t \in \tau(c)$. For every type $T_t \in \tau(c)$, choose a unique threshold $\alpha_t(c) \geq 0$ such that $\sum_{i \in N(c)} \max\{0, x_i^t(\theta) - \alpha_t(c)\} = \text{cost}(c)$. Each voter $i \in N(c)$ then pays $p_i^t(c) = \max\{0, x_i^t(\theta) - \alpha_t(c)\}$ units of type-T_t currency. After the payment, we update $x_i^t(\theta) \leftarrow x_i^t(\theta) - p_i^t(c)$ for every $i \in N(c)$ and every $T_t \in \tau(c)$. The balances of voters not approving c remain unchanged. The process stops when no unselected candidate is eligible.

While seqPhragmén offers weaker proportionality guarantees than MES, it has other attractive properties in the standard setting of participatory budgeting: in particular, it is *exhaustive*, in that, when it stops, no unselected candidate with at least one approver can be added to the committee without exceeding the budget. It is immediate from our description of the rule that τ-seqPhragmén is exhaustive in this sense as well: indeed, the rule does not stop if there is an eligible candidate.

6 Nested Types

We start by considering a class of constraint structures that generalizes MW-Single: it consists of constraint structures where the types are nested, i.e., ordered by inclusion.

Formally, we say that a constraint structure $\mathcal{H}$ is *nested* if (1) for every $T, T' \in \mathcal{H}$ it holds that either $T \subseteq T'$ or $T' \subseteq T$, and (2) there exists a type $T^\star \in \mathcal{H}$ such that $T^\star = C$.

A natural application of nested type constraints is a more granular version of repeated committee renewal. For example, one may impose an upper bound on the number of selected members who have served on the committee for at most r years. The corresponding candidate sets are nested by seniority, allowing turnover to be regulated across several experience levels rather than merely limiting replacements relative to the immediately preceding committee. More generally, nested constraints arise naturally when candidates or projects are classified by thresholds along a single ordered attribute, such as seniority, risk, qualification, or eligibility level.

It turns out that for τ-MW with nested types, both τ-MES and τ-seqPhragmén offer the same guarantees as in the standard setting. However, these results do not extend from τ-MW to τ-PB.

6.1 Nested Types: τ-MES

We first consider τ-MES in the context of τ-MW with nested types, and obtain a strong positive result.

Theorem 2. *In τ-MW with nested type constraints, τ-MES satisfies τ-EJR.*

Proof. Because the types are nested, we may relabel them so that $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_d$, with $T_1 = C$; let $k = b_1$.

Assume, for the sake of a contradiction, that τ-MES outputs a committee W that violates τ-EJR. Then there exists an integer $\ell \in [k]$ and an ℓ-deserving set of voters $S \subseteq N$ such that $|A_i \cap W| \leq \ell - 1$ for every $i \in S$. Let $X \subseteq \bigcap_{i \in S} A_i$ be a set witnessing that S is ℓ-deserving, so that $|X| \geq \ell$ and for each $t \in [d]$ the set $X_t = X \cap T_t$ satisfies $|X_t| \leq b_t \cdot \frac{|S|}{n}$. Since all candidates have unit cost, we may assume that $|X| = \ell$.

Set $X_{d+1} = \emptyset$ and let $Y_t = X_t \setminus X_{t+1}$ for each $t \in [d]$. Then $(Y_1, \ldots, Y_d)$ forms a partition of X, and for each $t \in [d]$ we have $Y_t \subseteq T_1, \ldots, T_t$.

Every voter in S approves all candidates in X, but $|A_i \cap W| \leq \ell - 1$ for all $i \in S$ and $|X| = \ell$, so $X \setminus W \neq \emptyset$. Let h be the smallest index such that $Y_h \setminus W \neq \emptyset$, and fix some candidate $x \in Y_h \setminus W$. By the choice of h, we have $Y_1 \cup \ldots \cup Y_{h-1} \subseteq W$.

Observe that for every $t \in [d]$ we have $Y_1 \cup \ldots \cup Y_{t-1} = X_1 \setminus X_t$ and hence $|Y_1| + \cdots + |Y_{t-1}| = \ell - |X_t|$. Therefore, if $t \leq h$, every voter in S approves at least $\ell - |X_t|$ candidates in $W \setminus T_t$.

We now prove the key claim.

Claim 1. *As long as x remains unselected, the remaining balance of every voter $i \in S$ in every type T_t with $t \in [h]$ is at least $\frac{1}{|S|}$.*

Proof of Claim 1. Suppose not, and let r be the first round after which there exists a voter $j \in S$ and an index $t \leq h$ such that $p_j^{(t)} < \frac{1}{|S|}$. By the choice of r, before round r every voter $i \in S$ has balance $p_i^{(t)} \geq \frac{1}{|S|}$ in each of the types T_t with $t \in [h]$.

Since $x \in Y_h \subseteq \bigcap_{i \in S} A_i$ and x belongs to types $T_1, \ldots, T_h$ only, the voters in S alone can afford x before round r by paying $\frac{1}{|S|}$ each in every relevant type. Hence, the price of x, i.e., $\max_{t \in [d]} \eta_t(x)$, is at most $\frac{1}{|S|}$ at that moment. Since the algorithm selects an affordable candidate of minimum price, the price of the candidate selected in round r does not exceed $\frac{1}{|S|}$.

Therefore, for each $t \in [d]$, in every round up to and including round r, each voter $j \in S$ pays at most $\frac{1}{|S|}$ in type T_t whenever she contributes towards a candidate in T_t. Thus, if q is the number of selected candidates in $A_j \cap T_t$ by the end of round r, then voter j has spent at most $\frac{q}{|S|}$ in type T_t.

On the other hand, after round r we have $p_j^{(t)} < \frac{1}{|S|}$, so voter j has spent more than $\frac{b_t}{n} - \frac{1}{|S|}$ in type T_t. Hence $\frac{q}{|S|} > \frac{b_t}{n} - \frac{1}{|S|}$, which implies $q > b_t \cdot \frac{|S|}{n} - 1$. Since q is an integer and $|X_t| \leq b_t \cdot \frac{|S|}{n}$, it follows that $q \geq |X_t|$.

However, we have argued that voter j also approves all $\ell - |X_t|$ candidates in $W \setminus T_t$. Therefore $|A_j \cap W| \geq (\ell - |X_t|) + q \geq \ell$, contradicting the assumption that $|A_j \cap W| \leq \ell - 1$. □

Since $x \notin W$, candidate x remains unselected until the algorithm terminates. By Claim 1, every voter in S still has balance at least $\frac{1}{|S|}$ in each of the types $T_1, \ldots, T_h$ at termination. As x is approved by all voters in S and belongs exactly to these types, the voters in S can still afford x at termination. This contradicts the stopping condition of the algorithm. Hence our assumption was false, and W satisfies τ-EJR. □

We will now show that this result is tight, in the sense that τ-MES no longer satisfies τ-EJR if the constraint structure is not nested (Proposition 5) or if the candidates have heterogeneous costs (Proposition 6).

Proposition 5. *There exists an instance of τ-MW with non-nested type constraints for which all outcomes of τ-MES fail to satisfy τ-EJR.*

Proof. Consider an instance of τ-MW with candidates $C = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ and voters $N = \{1, 2, 3, 4, 5, 6\}$. Let the type sets be $T_0 = C$, $T_1 = \{x_1, y_1\}$, $T_2 = \{x_2, y_2\}$, $T_3 = \{x_3, y_3\}$, with budgets $b_0 = 6$ and $b_1 = b_2 = b_3 = 2$. The family $\mathcal{H} = \{T_0, T_1, T_2, T_3\}$ is not nested, since T_1, T_2, T_3 are pairwise disjoint. The approval sets are $A_1 = \{x_1, x_2, x_3, y_1, z_1\}$, $A_2 = \{x_1, x_2, x_3, y_2, z_2\}$, $A_3 = \{x_1, x_2, x_3, y_3, z_3\}$, $A_4 = \{y_1, y_2, y_3, z_1\}$, $A_5 = \{y_1, y_2, y_3, z_2\}$, $A_6 = \{y_1, y_2, y_3, z_3\}$.

When we run τ-MES, initially each voter has balance 1 in type T_0 and $\frac{1}{3}$ in each of the types T_1, T_2, T_3. Each candidate y_i is approved by four voters and is priced at $\frac{1}{4}$, each candidate x_i is approved by three voters and is priced at $\frac{1}{3}$, and each candidate z_i is approved by two voters and is priced at $\frac{1}{2}$. Hence τ-MES first selects one of y_1, y_2, y_3.

Suppose y_1 is selected. Its four approvers each pay $\frac{1}{4}$ in types T_0 and T_1. In particular, voter 1 has remaining type-T_1 balance $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, while voters 2 and 3 retain balance $\frac{1}{3}$ each. Candidate x_1 is therefore no longer affordable, since the total remaining type-T_1 balance of its approvers $1, 2, 3$ is $\frac{1}{12} + \frac{1}{3} + \frac{1}{3} = \frac{3}{4} < 1$. By symmetry, selecting y_2 or y_3 makes x_2 or x_3 unaffordable in exactly the same way. Moreover, selecting any y_i does not affect the balances in types T_j, $j \in \{1, 2, 3\} \setminus \{i\}$. Thus τ-MES selects all of y_1, y_2, y_3, in some order, after which none of x_1, x_2, x_3 is affordable.

After y_1, y_2, y_3 have been selected, voters $1, 2, 3$ each have remaining type-T_0 balance $\frac{3}{4}$, while voters $4, 5, 6$ each have remaining type-T_0 balance $\frac{1}{4}$. Candidate z_1 is approved by voters 1 and 4, candidate z_2 by voters 2 and 5, and candidate z_3 by voters 3 and 6. In each case, the two approvers have total remaining type-T_0 balance 1, so all three candidates are affordable. Moreover, their pairs of approvers are disjoint, so selecting one of z_1, z_2, z_3 does not affect the affordability of the other two. Thus τ-MES selects all three and terminates with $W = \{y_1, y_2, y_3, z_1, z_2, z_3\}$.

Observe that $S = \{1, 2, 3\}$ is 3-deserving, as witnessed by $X = \{x_1, x_2, x_3\} \subseteq A_1 \cap A_2 \cap A_3$. Indeed, we have $|X| = 3$, $|X \cap T_0| = 3 \leq b_0 \cdot \frac{|S|}{|N|}$, and $|X \cap T_1| = |X \cap T_2| = |X \cap T_3| = 1 \leq 2 \cdot \frac{|S|}{|N|}$. However, each voter in S approves exactly two selected candidates: voter 1 approves y_1 and z_1, voter 2 approves y_2 and z_2, and voter 3 approves y_3 and z_3. Thus, no voter in S receives 3 approved candidates, even though S is 3-deserving. Therefore τ-MES may fail τ-EJR when the type family is not nested. Note that in our example $|W| = 6 = b_0$, so the global budget is exhausted. Hence no additional candidate can be added while preserving feasibility, and W is an inclusion-maximal feasible committee. □

We note that the constraint structure constructed in the proof of Proposition 5 is laminar (see Section 7), i.e., our negative result holds for laminar constraint structures.

Our next result shows that Theorem 2 cannot be extended from multiwinner voting to participatory budgeting.

Proposition 6. *There exists an instance of τ-PB with nested type constraints for which all outcomes of τ-MES fail to satisfy τ-EJR.*

Proof. Consider an instance with voters $N = \{1, 2, 3\}$ and candidates $C = \{x, x', y, z, z'\}$. Let $T_3 = \{x, x'\} \subset T_2 = \{x, x', y\} \subset T_1 = C$, so the type structure is nested. Let the type budgets be $b_1 = 33, b_2 = 12, b_3 = 12$. Let the costs of candidates be $\text{cost}(x) = \text{cost}(y) = \text{cost}(z) = 6$, $\text{cost}(x') = 8$, $\text{cost}(z') = 14$. Finally, the approval sets are $A_1 = \{x, x', y, z'\}$, $A_2 = \{x\}$, $A_3 = \{x', z, z'\}$.

When we run τ-MES, initially the voters' balances for types T_1, T_2 and T_3 are, respectively, 11, 4, and 4.

At the first step, candidate x has price 3, candidate x' has price 4, candidate z has price 6, and candidate z' has price 7. Candidate y is not affordable, since $\text{cost}(y) = 6$, but its only approver only has balance 4 in type T_2. Hence x is chosen first, and voters 1 and 2 each pay 3 in every type.

Candidate y remains unaffordable, and x' is no longer affordable because the total type-T_3 balance of its approvers $\{1, 3\}$ is $1 + 4 = 5 < 8$. Candidate z is affordable with price 6, while candidate z' is affordable with price 7. Thus z is selected second, and voter 3 pays 6 in type T_1.

At this point, the remaining type-T_1 balances of voters 1 and 3 are 8 and 5, respectively. Hence z' is no longer affordable, since $8 + 5 = 13 < 14$. Candidates y and x' remain unaffordable as well. Therefore, the algorithm terminates with $W = \{x, z\}$.

Observe that $S = \{1, 3\}$ is 2-deserving, as witnessed by $X = \{x', z'\} \subseteq A_1 \cap A_3$. Indeed, we have $\text{cost}(X \cap T_1) = 22 \leq 33 \cdot \frac{|S|}{|N|}$, $\text{cost}(X \cap T_2) = \text{cost}(X \cap T_3) = 8 \leq 12 \cdot \frac{|S|}{|N|}$. However, each voter in S approves exactly one selected candidate. Thus, τ-MES fails τ-EJR. □

6.2 Nested Types: τ-seqPhragmén

We next establish that τ-seqPhragmén satisfies τ-PJR for multiwinner voting with nested types.

Theorem 3. *In τ-MW with nested types, τ-seqPhragmén satisfies τ-PJR.*

Proof. Let W be the committee returned by τ-seqPhragmén, and let $\widehat{W} \subseteq W$ be the set of candidates selected by time 1. Note that by this time the total type-T_t currency given to the voters is $\frac{b_t}{n} \cdot n = b_t$, for each $T_t \in \mathcal{H}$.

Suppose for a contradiction that W violates τ-PJR, as witnessed by a group of voters $S \subseteq N$ and a witness set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| = \ell$. Let $U = \bigcup_{i \in S} A_i$. Then $|X_t| \leq b_t \cdot \frac{|S|}{n}$ for every $T_t \in \mathcal{H}$, and $|W \cap U| \leq \ell - 1$. Since $\widehat{W} \subseteq W$, we also have $|\widehat{W} \cap U| \leq \ell - 1$.

Relabel the types so that $C = T_1 \supseteq T_2 \supseteq \cdots \supseteq T_d$. Set $X_{d+1} = \emptyset$, and define $Y_t = X_t \setminus X_{t+1}$. Then $(Y_1, \ldots, Y_d)$ forms a partition of X, and for each $t \in [d]$ we have $Y_t \subseteq T_1, \ldots, T_t$.

Since $|X| = \ell$ and $X \subseteq U$, but $|\widehat{W} \cap U| < \ell$, we have $X \setminus \widehat{W} \neq \emptyset$. Let h be the smallest index such that $Y_h \setminus \widehat{W} \neq \emptyset$, and fix some candidate $x \in Y_h \setminus \widehat{W}$. We will argue that x is both affordable and eligible at time 1.

For affordability, note that, by our choice of h, we have $Y_1 \cup \cdots \cup Y_{h-1} \subseteq \widehat{W}$. Hence, for every $t \in [h]$ we have $X \setminus X_t = Y_1 \cup \cdots \cup Y_{t-1} \subseteq (\widehat{W} \cap U) \setminus T_t$ and $|X \setminus X_t| = \ell - |X_t|$. Therefore,

$$|\widehat{W} \cap U \cap T_t| = |\widehat{W} \cap U| - |(\widehat{W} \cap U) \setminus T_t| \leq (\ell - 1) - (\ell - |X_t|) = |X_t| - 1.$$

By time 1, voters in S have earned $|S| \cdot \frac{b_t}{n} \geq |X_t|$ units of type-T_t currency. They could have spent type-T_t currency on candidates in $\widehat{W} \cap U \cap T_t$ only, and each such candidate costs them at most one unit of type-T_t currency. Thus, at time 1, voters in S still hold at least one unit of type-T_t currency. Since this holds for every $t \in [h]$ and x belongs to types $T_1, \ldots, T_h$ only, the voters in S can jointly afford x at time 1.

For eligibility, we observe that for every $t \in [h]$ voters in S still hold a positive amount of type-T_t currency at time 1. Hence, the total amount of type-T_t currency spent by time 1 is strictly less than b_t and therefore $|\widehat{W}_t| \leq b_t - 1$. It follows that adding x would not violate any relevant type budget, i.e., x is eligible at time 1.

We conclude that x is both affordable and eligible at time 1, contradicting the definition of $\widehat{W}$.

Therefore $|\widehat{W} \cap U| \geq \ell$; since $\widehat{W} \subseteq W$, this implies $|W \cap U| \geq \ell$, contradicting the assumption that W violates τ-PJR. Hence, τ-seqPhragmén satisfies τ-PJR for τ-MW with nested types. □

Theorem 3 cannot be strengthened to τ-EJR. Indeed, in the special case of standard multiwinner voting, τ-EJR coincides with EJR and τ-seqPhragmén coincides with seqPhragmén, and it is known that seqPhragmén does not satisfy EJR [Brill et al., 2024]. Moreover, just as for τ-MES, Theorem 3 cannot be extended from multiwinner voting to participatory budgeting.

Proposition 7. *There exists an instance of τ-PB with nested type constraints for which all outcomes of τ-seqPhragmén fail to satisfy τ-PJR.*

Proof. Consider an instance with a single voter, so that $N = \{1\}$, and candidate set $C = \{x, y, z\}$. Let $T_3 = \{x\} \subset T_2 = \{x, y\} \subset T_1 = C$, so the type structure is nested. Let the type budgets be $b_1 = 12$, $b_2 = 8$, $b_3 = 4$. Let the costs of candidates be $\text{cost}(x) = 4$, $\text{cost}(y) = 5$, $\text{cost}(z) = 8$. The unique voter approves all candidates, so $A_1 = \{x, y, z\}$. When we run τ-seqPhragmén, voter 1 earns type-1, type-2 and type-3 currency at rates 12, 8 and 4, respectively.

Candidate x belongs to T_1, T_2 and T_3, so it becomes affordable at time $\max\{\frac{4}{12}, \frac{4}{8}, \frac{4}{4}\} = 1$. Candidate y belongs to T_1 and T_2, so it becomes affordable at time $\max\{\frac{5}{12}, \frac{5}{8}\} = \frac{5}{8}$. Candidate z belongs to T_1 only, so it becomes affordable at time $\frac{8}{12} = \frac{2}{3}$. Since $\frac{5}{8} < \frac{2}{3} < 1$, candidate y is the first candidate to become affordable. It is also eligible, as $\text{cost}(y) = 5 \leq \min\{b_1, b_2\}$. Hence y is selected first, at time $\frac{5}{8}$, and voter 1 pays 5 units of currency in each of the types T_1 and T_2.

After y is selected, we have $\text{cost}(W \cap T_1) = \text{cost}(W \cap T_2) = 5$. Candidate x is no longer eligible, since $\text{cost}(W \cap T_2) + \text{cost}(x) = 9 > 8 = b_2$. Candidate z is no longer eligible either, since $\text{cost}(W \cap T_1) + \text{cost}(z) = 13 > 12 = b_1$. Therefore, the algorithm terminates with $W = \{y\}$. No ties occur during this execution, so W is the unique outcome of τ-seqPhragmén on this instance.

Observe that $S = \{1\}$ is 2-deserving, as witnessed by $X = \{x, z\} \subseteq A_1$. Indeed, we have $\text{cost}(X \cap T_1) = 12 \leq 12 \cdot \frac{|S|}{|N|}$, $\text{cost}(X \cap T_2) = 4 \leq 8 \cdot \frac{|S|}{|N|}$, and $\text{cost}(X \cap T_3) = 4 \leq 4 \cdot \frac{|S|}{|N|}$. However, $W \cap A_1 = \{y\}$, so $\left|W \cap \bigcup_{i \in S} A_i\right| = 1 < 2$. Thus, τ-seqPhragmén fails τ-PJR. □

6.3 Nested Types: Thiele Rules

Finally, we consider Thiele rules. Given that PAV (which is a Thiele rule) satisfies EJR in the multiwinner setting without constraints, one may expect it to satisfy τ-EJR in τ-MW, perhaps under some constraints on the type structure. However, we will now show that this is not the case even for MW-Single, and even for the weaker τ-PJR axiom.

Proposition 8. *For each weight function f, there exists an instance of* MW-Single *on which the f-Thiele rule does not return a committee satisfying τ-PJR.*

Proof. Sánchez-Fernández et al. [2017] show that PAV is the only Thiele rule that satisfies PJR in the multiwinner setting without constraints. Since MW-Single generalizes this setting (if $b = k$, then every size-k committee is feasible and τ-PJR coincides with PJR), every Thiele rule other than PAV fails τ-PJR on MW-Single. Hence, it suffices to prove the claim for PAV.

Consider the following instance of MW-Single. Let $n = 14$, $k = 6$ and $b = 3$. Let $C = \{x, z\} \cup Y \cup D$, where $Y = \{y_1, y_2, y_3\}$ and $|D| = 5$. Consider $W_0 = \{z\} \cup D$.

The voters are partitioned into a group S of size 5 and three groups Q_1, Q_2, Q_3, each of size 3. Every voter in S approves exactly $\{x, z\}$. For each $j \in [3]$, every voter in Q_j approves exactly $\{y_j\}$. Note that candidates in D receive no approvals.

Let W be a feasible committee with the maximum PAV score. We claim that $z \in W$. Indeed, otherwise either $W \subseteq \{x\} \cup Y$ or $W \cap D \neq \emptyset$. In the former case, we have $|W| \leq 4$, so adding z to W increases the PAV score without violating feasibility (as $z \in W_0$). In the latter case we can replace some $d \in W$ with z; again, this increases the PAV score (since no voter approves d) without violating feasibility.

We will now argue that $x \notin W$. Indeed, suppose for the sake of contradiction that this is not the case. Since $|W \setminus W_0| \leq b$ and $|C \setminus W_0| = |\{x\} \cup Y| = b + 1$, there exists some $y_j \in Y$ such that $y_j \notin W$. Define $W' = (W \setminus \{x\}) \cup \{y_j\}$, and note that W' is also a feasible committee.

As $z \in W$, every voter in S approves exactly two candidates in W and exactly one candidate in W'. Thus removing x decreases the total PAV score by exactly $5 \cdot \frac{1}{2} = \frac{5}{2}$. On the other hand, adding y_j increases the total score by $|Q_j| = 3$. Since $3 > \frac{5}{2}$, the PAV score of W' exceeds that of W, contradicting the optimality of W. Hence no winning committee contains x.

Note, however, that S is 2-deserving, as witnessed by $X = \{x, z\} \subseteq \bigcap_{i \in S} A_i$. Indeed, $|X| = 2 \leq k \cdot \frac{|S|}{n} = \frac{15}{7}$ and $|X \setminus W_0| = 1 \leq b \cdot \frac{|S|}{n} = \frac{15}{14}$. Since the voters in S approve $\{x, z\}$ only, and we have argued that $x \notin W$, each voter in S only approves one candidate in W. It follows that PAV violates τ-PJR. □

6.4 Nested Types: Computational Hardness

The negative results for nested types so far concern specific voting rules. This naturally raises the question of whether some other polynomial-time rule can offer stronger guarantees. For τ-MW, τ-MES satisfies τ-EJR under nested type constraints (Theorem 2). For τ-PB, it still satisfies the weaker τ-PJR axiom, even for a broader class of laminar type constraints (see Section 7), but it need not satisfy τ-EJR (Proposition 6). We now show that this limitation is not specific to τ-MES.

Theorem 4. *Unless* P = NP*, no polynomial-time rule satisfies τ-EJR in τ-PB with nested type constraints.*

Proof. We reduce from the following PCP hardness result for k-uniform hypergraph vertex cover [Dinur et al., 2003]. There exists a fixed integer κ and constants $0 < \alpha < \beta < 1$ with $\beta - \alpha > \frac{1}{2}$ such that, given a κ-uniform hypergraph $G = (V, E)$, it is NP-hard to distinguish between the following two cases:

- G has a vertex cover of size at most $\alpha \cdot |V|$.
- Every vertex cover of G has size at least $\beta \cdot |V|$.

Choose a rational number γ with $\frac{1}{2} < \gamma = \frac{s}{q} < \beta - \alpha$, where $s, q \in \mathbb{N}^+$. Fix an integer $M \geq \frac{2\gamma}{2\gamma-1}$. Given a hypergraph $G = (V, E)$, create one voter for each vertex, so $n = |V|$. We can assume without loss of generality that $\gamma \cdot n$ is an integer. Otherwise, before constructing the voting instance, we can replace G by the disjoint union of q copies of G. For every hyperedge $e \in E$ and every vertex $v \in e$, create a candidate $c_{e,v}$. Voter u disapproves $c_{e,u}$ for each hyperedge e with $u \in e$ and approves all other candidates. Thus, each candidate is disapproved by exactly one voter.

Write $E = \{e_1, \ldots, e_m\}$. For each $j \in [m]$, let $L_j = \{c_{e_j,v} : v \in e_j\}$, and create a type $T_j = L_j \cup L_{j+1} \cup \cdots \cup L_m$. Then we have $C = T_1 \supseteq T_2 \supseteq \cdots \supseteq T_m$, and hence the constraint structure is nested. Every candidate in L_j has cost M^{m-j}, and type T_j has budget $b_j = \frac{1}{\gamma} \sum_{f=j}^{m} M^{m-f}$.

We first observe that every feasible outcome selects at most one candidate from each set L_j. Indeed, for every feasible outcome W and every $j \in [m]$ we have

$$\text{cost}(W \cap L_j) \leq \text{cost}(W \cap T_j) \leq b_j = \frac{M^{m-j}}{\gamma} \sum_{f=0}^{m-j} M^{-f} < \frac{M^{m-j}}{\gamma} \cdot \frac{M}{M-1} \leq 2M^{m-j},$$

where the last inequality follows from $M \geq \frac{2\gamma}{2\gamma-1}$. Thus W can contain at most one candidate of L_j. Accordingly, every feasible outcome has size at most $|E|$.

Now suppose a polynomial-time rule always outputs a feasible outcome satisfying τ-EJR. Given G, construct the instance above and run the rule to obtain W. Accept if and only if W selects exactly one candidate from every set L_j, $j \in [m]$, and more than $(1-\beta)n$ voters approve all candidates in W. Below, we show that every τ-EJR outcome passes this test if G admits a vertex cover of size at most $\alpha \cdot n$, whereas no τ-EJR outcome passes it if every vertex cover of G is of size at least $\beta \cdot n$. Thus, any polynomial-time rule that always returns a τ-EJR outcome can be used to distinguish these two cases of the underlying hypergraph vertex cover problem.

Suppose first that G has a vertex cover $V^\star \subseteq V$ with $|V^\star| \leq \alpha \cdot |V| = \alpha \cdot n$. For each edge $e \in E$, fix exactly one representative $v_e \in V^\star \cap e$. Define $X = \{c_{e,v_e} : e \in E\}$. Note that $|X| = |E|$ and that X contains exactly one candidate from each set L_j. Let $V^- = V \setminus V^\star$. Then $|V^-| \geq (1-\alpha) \cdot n$, and every voter in V^- approves all candidates in X. Let W be a feasible outcome satisfying τ-EJR, and let $S = \{u \in V^- : |W \cap A_u| < |E|\}$.

We claim that $|S| \leq \gamma \cdot n - 1$. Indeed, otherwise $|S| \geq \gamma \cdot n$ (by integrality of $\gamma \cdot n$), and hence the pair (S, X) would witness a violation of τ-EJR. Specifically, we have $\text{cost}(X \cap T_j) = \sum_{f=j}^{m} M^{m-f} = \gamma \cdot b_j \leq b_j \cdot \frac{|S|}{n}$ for each type T_j, all voters in S approve all candidates in X, and $|X| = |E|$, but no voter in S approves $|E|$ candidates in W, contradicting τ-EJR.

Hence,

$$|V^- \setminus S| \geq |V^-| - \gamma \cdot n + 1 \geq (1 - \alpha - \gamma) \cdot n + 1 > (1 - \beta) \cdot n.$$

Moreover, since all voters in $V^- \setminus S$ approve at least $|E|$ candidates in W, and $|W| \leq |E|$, it has to be the case that W selects exactly one candidate from each set L_j, $j \in [m]$, and, moreover, all voters in $V^- \setminus S$ approve all candidates in W. Thus, W passes our check.

Conversely, suppose every vertex cover for G has size at least $\beta \cdot n$. Consider a feasible outcome W. We will argue that W fails our check. By our first observation, W selects at most one candidate of each set L_j , and if $W \cap L_j = \emptyset$ for some $j \in [m]$, then W fails our check. Thus, assume that W contains exactly one candidate of each set L_j. Then the set $V' = \{v \in V : c_{e,v} \in W \text{ for some } e \in E\}$ forms a vertex cover, and hence its size is at least $\beta \cdot n$. Note that each voter in V' disapproves at least one candidate in W, and hence at most $|V \setminus V'| \leq (1-\beta) \cdot n$ voters approve all candidates in W, which fails the second condition of the verification procedure. □

7 Laminar Types

Given our positive results for nested types, it is natural to ask if they can be extended to a broader class of type constraints. In this section, we consider a natural extension of nested types, namely, laminar types. We show that, for τ-MES, we can still obtain non-trivial (but weaker) proportionality

guarantees in this more general setting. For τ-seqPhragmén, our positive result also extends to this setting, but it breaks down once the types are allowed to be non-laminar.

A constraint structure $\mathcal{H}$ is *laminar* [Schrijver, 2003] if, for every $T, T' \in \mathcal{H}$ with $T \cap T' \neq \emptyset$ either $T \subseteq T'$ or $T' \subseteq T$. Observe that every nested constraint structure is laminar, but the converse is not true. An instance of τ-PB is said to be *laminar* if its constraint structure is laminar. This definition naturally captures hierarchical type structures such as, for example, spending limits both for neighborhoods and for candidate categories and sub-categories within those neighborhoods[2]. Laminar candidate groupings have been previously studied in the context of PB [Goyal et al., 2023] and multiwinner voting [Bredereck et al., 2018].

7.1 Laminar Types: τ-MES

On the positive side, τ-MES guarantees τ-PJR for laminar types; this result holds even for PB, i.e., for arbitrary candidate costs.

Theorem 5. *In τ-PB with laminar type constraints, τ-MES satisfies τ-PJR.*

Proof. Fix $\ell \geq 1$ and an ℓ-deserving group $S \subseteq N$. By definition, there exists a set $X \subseteq \bigcap_{i \in S} A_i$ such that $|X| \geq \ell$ and $\text{cost}(X \cap T_t) \leq b_t \cdot \frac{|S|}{n}$ for every $t \in [d]$. By discarding extra candidates, we may assume that $|X| = \ell$. For simplicity, write $X_t = X \cap T_t$.

Suppose towards a contradiction that $\left|W \cap \bigcup_{i \in S} A_i\right| < \ell$. Let $y_1, \ldots, y_r$ be the candidates in $W \cap \bigcup_{i \in S} A_i$, listed in the order in which τ-MES selects them. Then $r < \ell$.

For each $j \in \{0, \ldots, r\}$, each $i \in S$, and each $t \in [d]$, let $q_{i,j}^{(t)}$ be voter i's remaining type-t balance immediately after y_j is selected, with the convention $q_{i,0}^{(t)} = \frac{b_t}{n}$. Since $y_1, \ldots, y_r$ are exactly the selected candidates approved by at least one voter in S, any candidate that is selected after y_{j-1}, but before y_j is approved by no voter in S, and hence does not change these balances.

We define the residual witness sets $R_r \subseteq \cdots \subseteq R_1 \subseteq R_0 \subseteq X$ recursively. Intuitively, for each $j \in \{0, \ldots, r\}$, the set R_j consists of the candidates in the original witness set X that have not yet been matched to the selected candidates $y_1, \ldots, y_j$. We start with $R_0 = X$. Now fix $j \in \{1, \ldots, r\}$, and define R_j from R_{j-1} as follows.

- If $y_j \in R_{j-1}$, set $x_j = y_j$ and define $R_j = R_{j-1} \setminus \{x_j\}$.
- If $y_j \notin R_{j-1}$, define $\Lambda_j = \{\, t : y_j \in T_t \text{ and } R_{j-1} \cap T_t \neq \emptyset \,\}$. Thus, Λ_j is the set of indices of types that contain y_j and intersect the residual set R_{j-1}.
  - If $\Lambda_j = \emptyset$, set $R_j = R_{j-1}$.
  - If $\Lambda_j \neq \emptyset$, then every set T_t with $t \in \Lambda_j$ contains y_j. Since $\mathcal{H}$ is laminar, these sets form a chain under inclusion. Let $u_j \in \Lambda_j$ be such that $T_{u_j} \subseteq T_t$ for every $t \in \Lambda_j$. Choose any $x_j \in R_{j-1} \cap T_{u_j}$, and define $R_j = R_{j-1} \setminus \{x_j\}$.

For each $j \in \{0, \ldots, r\}$ and each type $T_t \in \mathcal{H}$, write $(R_j)_t = R_j \cap T_t$.

Claim 2. *For every $j \in \{0, \ldots, r\}$, the following properties hold:*

(i) $R_j \cap \{y_1, \ldots, y_j\} = \emptyset$

(ii) $|R_j| \geq \ell - j$

(iii) for every $i \in S$ and every $t \in [d]$, $q_{i,j}^{(t)} \geq \frac{\text{cost}((R_j)_t)}{|S|}$

[2]Here, neighborhood budgets are viewed as spending limits on a citywide resource shared by the full electorate, rather than separate budgets controlled by the residents of each neighborhood.

Proof of Claim 2. We proceed by induction on j. For $j = 0$, (i) and (ii) are immediate from $R_0 = X$ and $|X| = \ell$. Also, since $\text{cost}(X_t) \le b_t \cdot \frac{|S|}{n}$, we have $q_{i,0}^{(t)} = \frac{b_t}{n} \ge \frac{\text{cost}(X_t)}{|S|} = \frac{\text{cost}((R_0)_t)}{|S|}$, so (iii) holds.

Now assume the claim holds for $j-1$. Properties (i) and (ii) follow directly from the construction of R_j. At step j we remove at most one candidate, and any removed candidate lies in R_{j-1}, hence is not among $y_1, \dots, y_{j-1}$ by (i) for $j-1$. It remains to prove property (iii). We proceed by considering the three cases in the definition of R_j. In each case we make use of the fact that no voter in S contributes to buying any candidate between y_{j-1} and y_j; hence, we only have to reason about the change in the remaining budgets following the purchase of y_j.

Case 1: $y_j \in R_{j-1}$.

In this case, by definition, $x_j = y_j$ and $R_j = R_{j-1} \setminus \{y_j\}$. Since $y_j \in R_{j-1} \subseteq X \subseteq \bigcap_{i \in S} A_i$, every voter in S approves y_j.

Fix a type $t \in [d]$. If $y_j \notin T_t$, then $\eta_t(y_j) = 0$ by definition. Thus, type-t balances are not changed when passing from R_{j-1} to R_j. Therefore $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$ and $(R_j)_t = (R_{j-1})_t$ and property (iii) follows by the induction hypothesis.

If $y_j \in T_t$, then $y_j \in (R_{j-1})_t$, and so property (iii) for $j-1$ gives $q_{i,j-1}^{(t)} \ge \frac{\text{cost}((R_{j-1})_t)}{|S|} \ge \frac{\text{cost}(y_j)}{|S|}$ for all $i \in S$. Thus, since the voters in S all approve y_j, they could pay for y_j alone by each paying at most $\frac{\text{cost}(y_j)}{|S|}$ in type t. Therefore $\eta_t(y_j) \le \frac{\text{cost}(y_j)}{|S|}$ for every $t \in [d]$. Hence, we have $q_{i,j}^{(t)} \ge q_{i,j-1}^{(t)} - \eta_t(y_j) \ge \frac{\text{cost}((R_{j-1})_t)}{|S|} - \frac{\text{cost}(x_j)}{|S|} = \frac{\text{cost}((R_j)_t)}{|S|}$, where the second inequality uses property (iii) for $j-1$, and the equality uses $x_j \in T_t$ together with $R_j = R_{j-1} \setminus \{y_j\}$. Thus, property (iii) holds.

Case 2: $y_j \notin R_{j-1}$ and $\Lambda_j = \emptyset$.

In this case, by definition, $R_j = R_{j-1}$. Fix $t \in [d]$. If $y_j \notin T_t$, then type t is not charged at step j, so $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$. Since also $R_j = R_{j-1}$, property (iii) follows immediately from the induction hypothesis.

If $y_j \in T_t$, then the definition of Λ_j and the assumption $\Lambda_j = \emptyset$ imply that $R_{j-1} \cap T_t = \emptyset$. Since $R_j = R_{j-1}$, we also have $(R_j)_t = \emptyset$. Therefore $q_{i,j}^{(t)} \ge 0 = \frac{\text{cost}((R_j)_t)}{|S|}$ and property (iii) holds.

Case 3: $y_j \notin R_{j-1}$ and $\Lambda_j \neq \emptyset$.

In this case, by definition, we choose $u_j \in \Lambda_j$ such that $T_{u_j} \subseteq T_t$ for every $t \in \Lambda_j$, then choose $x_j \in R_{j-1} \cap T_{u_j}$, and finally set $R_j = R_{j-1} \setminus \{x_j\}$. Since $x_j \in R_{j-1} \subseteq X$ and $X \subseteq \bigcap_{i \in S} A_i$, every voter in S approves x_j.

Consider any type $u \in [d]$. If $x_j \notin T_u$, then $\eta_u(x_j) = 0$ by definition. If $x_j \in T_u$, then $x_j \in (R_{j-1})_u$, so property (iii) for $j-1$ gives $q_{i,j-1}^{(u)} \ge \frac{\text{cost}((R_{j-1})_u)}{|S|} \ge \frac{\text{cost}(x_j)}{|S|}$ for all $i \in S$. Since every voter in S approves x_j, the voters in S can pay for x_j alone by each paying at most $\frac{\text{cost}(x_j)}{|S|}$ in type u.

Since this holds for any type, we have $\eta_u(x_j) \le \frac{\text{cost}(x_j)}{|S|}$ for every $u \in [d]$. Since $x_j \in R_{j-1}$ while $y_j \notin R_{j-1}$, we know $x_j \neq y_j$. Hence, by property (i) for $j-1$, x_j was still unselected when y_j was chosen. Since x_j was affordable and unselected at that moment, and the algorithm chooses y_j so as to minimize the price among all affordable unselected candidates, we conclude that $\max_{u \in [d]} \eta_u(y_j) \le \max_{u \in [d]} \eta_u(x_j) \le \frac{\text{cost}(x_j)}{|S|}$.

Now fix $t \in [d]$. If $x_j \in T_t$, then using the above inequality we have

$$q_{i,j}^{(t)} \ge q_{i,j-1}^{(t)} - \eta_t(y_j) \ge \frac{\text{cost}((R_{j-1})_t)}{|S|} - \frac{\text{cost}(x_j)}{|S|} = \frac{\text{cost}((R_j)_t)}{|S|}.$$

Assume next that $x_j \notin T_t$. If $y_j \notin T_t$, then type t is not charged at step j, and since also $x_j \notin T_t$, removing x_j does not change the type-t part of the residual set. Thus $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$ and $(R_j)_t = (R_{j-1})_t$, so property (iii) follows immediately from the induction hypothesis.

Finally, suppose that $y_j \in T_t$. We claim that $R_{j-1} \cap T_t = \emptyset$. Indeed, if $R_{j-1} \cap T_t \neq \emptyset$, then by the definition of Λ_j we would have $t \in \Lambda_j$, because $y_j \in T_t$ and $R_{j-1} \cap T_t \neq \emptyset$. Since u_j was chosen so that $T_{u_j} \subseteq T_{t'}$ for every $t' \in \Lambda_j$, it would follow in particular that $T_{u_j} \subseteq T_t$. But $x_j \in T_{u_j}$ by construction, so this would imply $x_j \in T_t$, contradicting the assumption of the present case. Hence $R_{j-1} \cap T_t = \emptyset$. Since also $x_j \notin T_t$, removing x_j does not change the type-t part of the residual set,

and therefore $(R_j)_t = \emptyset$. Consequently, $q_{i,j}^{(t)} \geq 0 = \frac{\text{cost}((R_j)_t)}{|S|}$. Thus property (iii) holds in Case 3 as well.

This completes the induction for property (iii), and thus the proof of the claim. □

By Claim 2 (ii), for $j = r$, we have $|R_r| \geq \ell - r > 0$. Choose any $x \in R_r$. Since $R_r \subseteq X \subseteq \bigcap_{i \in S} A_i$, every voter in S approves x. We claim that $x \notin W$. Indeed, if $x \in W$, then $x \in W \cap \bigcup_{i \in S} A_i$, so by the definition of $y_1, \ldots, y_r$ we would have $x \in \{y_1, \ldots, y_r\}$, contradicting Claim 2 (i) for $j = r$.

Let $\bar{p}_i^{(t)}$ denote voter i's final type-t balance when the algorithm terminates. Since no candidate selected after y_r is approved by any voter in S, the balances of voters in S never change after the selection of y_r, and thus $\bar{p}_i^{(t)} = q_{i,r}^{(t)}$ for every $i \in S$ and $t \in [d]$. If x belongs to no type, then $\eta_t(x) = 0$ for every $t \in [d]$, so x is affordable at termination, contradicting the stopping condition.

Thus fix any $t \in [d]$ such that $x \in T_t$. Because $x \in (R_r)_t$, Claim 2 (iii) for $j = r$ gives $q_{i,r}^{(t)} = \bar{p}_i^{(t)} \geq \frac{\text{cost}((R_r)_t)}{|S|} \geq \frac{\text{cost}(x)}{|S|}$ for all $i \in S$. Since every voter in S approves x, we have $\sum_{i:\, x \in A_i} \min\left\{\frac{\text{cost}(x)}{|S|}, \bar{p}_i^{(t)}\right\} \geq \sum_{i \in S} \min\left\{\frac{\text{cost}(x)}{|S|}, \bar{p}_i^{(t)}\right\} = \sum_{i \in S} \frac{\text{cost}(x)}{|S|} = \text{cost}(x)$.

Thus $\Psi_t(x) \neq \emptyset$. Since t was arbitrary among the types containing x, the same argument shows that $\eta_t(x) \leq \frac{\text{cost}(x)}{|S|}$ for every type T_t with $x \in T_t$. Hence $x \in C^+$ when the algorithm terminates, contradicting the stopping condition.

This contradiction shows that $\left|W \cap \bigcup_{i \in S} A_i\right| \geq \ell$, and thus W satisfies τ-PJR. □

Theorem 4 already shows that no polynomial-time rule satisfies τ-EJR in τ-PB under nested (and hence laminar or more general) type constraints, unless $P = NP$. A natural next question is therefore whether τ-MES can still guarantee the weaker τ-PJR axiom beyond laminar type constraints. We answer this question negatively by showing that any non-laminar constraint structure can be embedded in an instance of τ-PB such that τ-MES violates τ-PJR on the resulting instance.

Proposition 9. *Let C be a candidate set, and let $\mathcal{H}$ be a non-laminar constraint structure on C. Then there exists an instance $\mathcal{I} = (N, C, \text{cost}, (A_i)_{i \in N}, d, \mathcal{H}, (b_t)_{t \in [d]})$ of τ-PB for which the outcome returned by τ-MES does not satisfy τ-PJR.*

Proof. Since $\mathcal{H}$ is not laminar, there exist two types $T_1, T_2 \in \mathcal{H}$ and candidates x, y, z such that $x \in T_1 \setminus T_2$, $y \in T_2 \setminus T_1$, $z \in T_1 \cap T_2$. Let $D = C \setminus \{x, y, z\}$. Set the candidate costs as

$$\text{cost}(x) = \text{cost}(y) = \frac{2}{5}, \quad \text{cost}(z) = \frac{1}{2}, \quad \text{cost}(w) = 1 \text{ for all } w \in D.$$

Set the type budgets as follows: $b_1 = b_2 = 1$, $b_r = 2$ for $r = 3, \ldots, d$. Let $N = \{1, 2, 3, 4, 5\}$. Define the approval sets by

$$A_1 = A_2 = \{x, y, z\}, \quad A_3 = A_4 = A_5 = \{z\}.$$

We first show that the group $S = \{1, 2\}$ is 2-deserving. Let $X = \{x, y\}$. Then $X \subseteq A_1 \cap A_2$ and $|X| = 2$. Further, we have

$$\text{cost}(X \cap T_1) = \text{cost}(x) = \frac{2}{5} = b_1 \cdot \frac{|S|}{n} \qquad \text{and} \qquad \text{cost}(X \cap T_2) = \text{cost}(y) = \frac{2}{5} = b_2 \cdot \frac{|S|}{n}.$$

Finally, for $r \in [d] \setminus \{1, 2\}$, we have $\text{cost}(X \cap T_r) \leq \text{cost}(x) + \text{cost}(y) = \frac{4}{5} = b_r \cdot \frac{|S|}{n}$. Thus, S is 2-deserving.

We now analyze the execution of τ-MES. Initially, every voter has type-1 and type-2 balance $\frac{1}{5}$, and balance $\frac{2}{5}$ for every other type.

Candidate z is approved by all five voters and belongs to both T_1 and T_2. Thus, for each type its cost can be covered by charging each voter $\frac{1}{10}$. Hence, $\max_{r \in [d]} \eta_r(z) = \frac{1}{10}$.

In contrast, candidates x and y are approved by voters 1 and 2 only, and $x \in T_1$, $y \in T_2$. Thus, we have $\eta_1(x) = \eta_2(y) = \frac{1}{5}$ and consequently $\max_{t \in [d]} \eta_t(x) = \max_{t \in [d]} \eta_t(y) = \frac{1}{5}$.

Now, consider a candidate $w \in D$; note that w receives no approvals. Thus if $w \in T$ for some $T \in \mathcal{H}$, then w is not affordable, and if w does not belong to any type, it will be added to W at the

start of the algorithm (indeed, note that w meets the condition to be added to C^+ in line 8 with $\eta_t(w) = 0$ for each $t \in [d]$) and not affect any voter's balance for any type.

We conclude that τ-MES selects z before selecting either x or y.

After z is selected, each voter has paid $\frac{1}{10}$ from her type-1 balance and $\frac{1}{10}$ from her type-2 balance. Therefore the remaining type-1 balance of voters 1 and 2 is $\frac{1}{5} - \frac{1}{10} = \frac{1}{10}$. Since $\text{cost}(x) = \text{cost}(y) = \frac{2}{5}$, and voters 1 and 2 are the only ones who approve x and y, neither x nor y is affordable at that point. Hence, the winning committee will only contain a single candidate approved by any of the voters in S; since S is 2-deserving, we conclude that τ-MES violates τ-PJR on this instance. □

7.2 Laminar Types: τ-seqPhragmén

We have seen in Section 6 that τ-MES satisfies τ-EJR for τ-MW with nested types while the guarantee does not further extend to τ-MW with laminar types. In contrast, we now show that the guarantee of τ-seqPhragmén for τ-MW also extends to laminar types.

Theorem 6. *In τ-MW with laminar types, τ-seqPhragmén satisfies τ-PJR.*

Proof. Let W be the committee returned by τ-seqPhragmén, and let $\widehat{W} \subseteq W$ be the set of candidates selected by time 1. Note that by this time the total type-t currency given to the voters is $\frac{b_t}{n} \cdot n = b_t$.

Suppose for a contradiction that W violates τ-PJR. Then there exist an integer $\ell \geq 1$, a group $S \subseteq N$, and a witness set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| = \ell$ and $|X_t| \leq b_t \cdot \frac{|S|}{n}$ for every $t \in [d]$, where $X_t = X \cap T_t$, such that $|W \cap U| < \ell$, where $U = \bigcup_{i \in S} A_i$. Let $Z = \widehat{W} \cap U$. Since $\widehat{W} \subseteq W$, we have $|Z| \leq \ell - 1$. Note also that $X \subseteq U$, and hence $X \setminus Z = X \setminus \widehat{W}$.

We first show that every candidate $x \in X \setminus Z$ belongs to some type T_t with $|Z \cap T_t| \geq |X_t|$. Suppose not, i.e., $|Z \cap T_t| \leq |X_t| - 1$ for every type T_t that contains x, and fix such a type T_t. By time 1, voters in S have earned $|S| \cdot \frac{b_t}{n} \geq |X_t|$ units of type-t currency. They have spent type-t currency only on candidates in $Z \cap T_t$, and each such candidate costs at most one unit of type-t currency from S. Thus, at time 1, voters in S still hold at least $|X_t| - |Z \cap T_t| \geq 1$ unit of type-t currency. Consequently, the total type-t spending by time 1 is at most $b_t - 1$, so $|\widehat{W} \cap T_t| \leq b_t - 1$. Since this holds for every type containing x, and every voter in S approves x, candidate x is both affordable and eligible at time 1, contradicting the definition of $\widehat{W}$.

Let B be the union of all types $T_t \in \mathcal{H}$ with $|Z \cap T_t| \geq |X_t|$. Since $\mathcal{H}$ is laminar, any two of these types are either disjoint or nested. Consequently, we can write B as the disjoint union of the inclusion-maximal ones among them, i.e., $B = T_{u_1} \cup \cdots \cup T_{u_r}$, where $T_{u_j} \cap T_{u_{j'}} = \emptyset$ for all distinct $j, j' \in [r]$. Each of these maximal types T_{u_j} satisfies $|Z \cap T_{u_j}| \geq |X_{u_j}|$, and summing over them gives $|Z \cap B| \geq |X \cap B|$. Moreover, by the above, every candidate in $X \setminus Z$ lies in B, and thus $X \setminus B \subseteq Z \setminus B$. Therefore, we have

$$|Z| = |Z \cap B| + |Z \setminus B| \;\geq\; |X \cap B| + |X \setminus B| = |X| = \ell,$$

contradicting $|Z| \leq \ell - 1$. Hence, τ-seqPhragmén satisfies τ-PJR for τ-MW with laminar types. □

A natural question is whether the positive results for τ-MW extend to more general type structures. Below, we show that our analysis is tight as τ-seqPhragmén may violate τ-PJR for non-laminar types.

Proposition 10. *There exists an instance of τ-MW with non-laminar type constraints for which τ-seqPhragmén does not satisfy τ-PJR.*

Proof. Consider an instance of τ-MW with voters $N = \{1, 2, 3, 4\}$ and candidates $C = \{x, y, y', z\}$. Let $T_1 = C$, $T_2 = \{x, y, y'\}$ and $T_3 = \{y, y', z\}$, so the type structure is not laminar. Let the type budgets be $b_1 = 4$ and $b_2 = b_3 = 2$. Finally, the approval sets are $A_1 = A_2 = \{x, y, z\}$, $A_3 = \{y, y'\}$, $A_4 = \{y'\}$.

Each voter earns type-1 currency at rate 1 and type-2 and type-3 currency at rate $\frac{1}{2}$.

Candidate y is approved by voters 1, 2 and 3, who jointly earn type-1 currency at rate 3 and type-2 and type-3 currency at rate $\frac{3}{2}$. Hence y becomes affordable at time $\max\{\frac{1}{3}, \frac{2}{3}\} = \frac{2}{3}$. Each of

x, z and y' has two approvers, who jointly earn type-2 and type-3 currency at rate 1, and hence none of them is affordable before time 1. Thus y is selected first, at time $\frac{2}{3}$, and voters 1, 2 and 3 spend their entire type-2 and type-3 balances of $\frac{1}{3}$ on it.

Afterwards, voters 1 and 2 start again from zero in types 2 and 3, so x and z cannot become affordable before time $\frac{2}{3} + 1 = \frac{5}{3}$. In contrast, voter 4 still holds $\frac{1}{3}$ unit of type-2 and of type-3 currency, and voters 3 and 4 jointly hold at least one unit of type-1 currency since time $\frac{2}{3}$. Hence they can afford y' at time $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$. As T_2 and T_3 contain only y at that point, y' is eligible and is selected at time $\frac{4}{3}$. Now $|W \cap T_2| = |W \cap T_3| = 2$, so neither x nor z is eligible, and the algorithm terminates with $W = \{y, y'\}$. No ties occur, so W is the unique outcome.

Observe that $S = \{1, 2\}$ is 2-deserving, as witnessed by $X = \{x, z\} \subseteq A_1 \cap A_2$, because $|X \cap T_1| = 2 \leq 4 \cdot \frac{|S|}{|N|}$ and $|X \cap T_2| = |X \cap T_3| = 1 \leq 2 \cdot \frac{|S|}{|N|}$. However, $W \cap (A_1 \cup A_2) = \{y\}$. Thus, τ-seqPhragmén does not satisfy τ-PJR. □

Our results illustrate that moving from nested to laminar types affects the two rules differently. τ-MES loses its τ-EJR guarantee for τ-MW though it still satisfies τ-PJR for τ-PB. In contrast, τ-seqPhragmén retains its τ-PJR guarantee for τ-MW, and this guarantee does not extend beyond laminar types.

7.3 Laminar Types: Computational Hardness

By Theorem 4, no polynomial-time rule satisfies τ-EJR in τ-PB under nested (and hence laminar) type constraints, unless $P = NP$. This does not rule out the possibility of a polynomial-time rule that satisfies τ-EJR for τ-PB under pairwise disjoint types (another special case of laminar types). Moreover, the reduction in Theorem 4 relies on heterogeneous candidate costs. However, we show next that the hardness persists even for pairwise disjoint types and unit costs.

Theorem 7. *Unless* P = NP*, no polynomial-time rule satisfies τ-EJR in τ-PB with laminar type constraints, even in the special case of unit costs*[3] *and pairwise disjoint types.*

Proof. We reduce from the following PCP hardness result for k-uniform hypergraph vertex cover [Dinur et al., 2003]. [4] There exists a fixed integer κ and constants $0 < \alpha < \beta < 1$ with $\beta - \alpha > \frac{1}{2}$ such that, given a κ-uniform hypergraph $G = (V, E)$, it is NP-hard to distinguish between the following two cases:

- G has a vertex cover of size at most $\alpha \cdot |V|$.
- Every vertex cover of G has size at least $\beta \cdot |V|$.

Choose a rational number γ with $\frac{1}{2} < \gamma = \frac{s}{q} < \beta - \alpha$, where $s, q \in \mathbb{N}^+$. Given a hypergraph $G = (V, E)$, create one voter for each vertex, so $n = |V|$. For every hyperedge $e \in E$ and every vertex $v \in e$, create a candidate $c_{e,v}$. Voter u disapproves $c_{e,u}$ for each hyperedge e with $u \in e$ and approves all other candidates; thus, each candidate is disapproved by exactly one voter.

For each hyperedge $e \in E$, create a type $T_e = \{c_{e,v} : v \in e\}$. These types are pairwise disjoint, hence laminar. All candidates have unit cost, and each type has budget $b_e = \frac{1}{\gamma}$. Since $\gamma > \frac{1}{2}$, we have $\frac{1}{\gamma} < 2$, so every feasible outcome selects at most one candidate from each type. Accordingly, every feasible outcome has size at most $|E|$. We can assume without loss of generality that $\gamma \cdot n$ is an integer. Indeed, we have $\gamma = \frac{s}{q}$ with $s, q \in \mathbb{N}^+$; to ensure that q divides n, before constructing the voting instance, we can replace G by the disjoint union of q copies of G.

[3] We note that we reduce to instances that are more general than multiwinner voting since we allow for non-integral type budgets. Thus, this theorem does not rule out the possibility of a polynomial-time rule that satisfies τ-EJR in multiwinner voting with laminar type constraints.

[4] This reduction construction was discovered with the assistance of ChatGPT 5.5. The authors independently verified the resulting proof and refined its presentation. The authors take full responsibility for the correctness of the proof.

Now suppose a polynomial-time rule always outputs a feasible outcome satisfying τ-EJR. Given G, construct the instance above and run the rule to obtain W. Accept if and only if W selects exactly one candidate from every type T_e, $e \in E$, and more than $(1-\beta)n$ voters approve all candidates in W. Below, we show that every τ-EJR outcome passes this test if G admits a vertex cover of size at most $\alpha \cdot n$, whereas no τ-EJR outcome passes it if every vertex cover of G is of size at least $\beta \cdot n$. Thus, any polynomial-time rule that always returns a τ-EJR outcome can be used to distinguish these two cases of the underlying hypergraph vertex cover problem.

Suppose first that G has a vertex cover $V^\star \subseteq V$ with $|V^\star| \leq \alpha \cdot |V| = \alpha \cdot n$. For each edge $e \in E$, fix exactly one representative $v_e \in V^\star \cap e$. Define $X = \{c_{e,v_e} : e \in E\}$; note that $|X| = |E|$. Let $V^- = V \setminus V^\star$. Then $|V^-| \geq (1-\alpha) \cdot n$, and every voter in V^- approves all candidates in X. Let W be a feasible outcome satisfying τ-EJR, and let $S = \{u \in V^- : |W \cap A_u| < |E|\}$.

We claim that $|S| \leq \gamma \cdot n - 1$. Indeed, otherwise $|S| \geq \gamma \cdot n$ (by integrality of $\gamma \cdot n$), and hence the pair (S, X) would witness a violation of τ-EJR: we have $b_e \cdot \frac{|S|}{n} \geq \frac{1}{\gamma} \cdot \frac{\gamma \cdot n}{n} = 1 = |X \cap T_e|$ for each type T_e, all voters in S approve all candidates in X, $|X| = |E|$, but no voter in S approves $|E|$ candidates in W, contradicting τ-EJR.

Hence,

$$|V^- \setminus S| \geq |V^-| - \gamma \cdot n + 1 \geq (1 - \alpha - \gamma) \cdot n + 1 > (1-\beta) \cdot n.$$

Moreover, since all voters in $V^- \setminus S$ approve at least $|E|$ candidates in W, and $|W| \leq |E|$, it has to be the case that W selects exactly one candidate from each type T_e, $e \in E$, and, moreover, all voters in $V^- \setminus S$ approve all candidates in W, i.e., W passes our check.

Conversely, suppose every vertex cover for G has size at least $\beta \cdot n$. Consider a feasible outcome W; we will argue that W fails our check. By feasibility, W selects at most one candidate of each type, and if $W \cap T_e = \emptyset$ for some $e \in E$, then W fails our check. Thus, assume that W contains exactly one candidate of each type. Then the set $V' = \{v \in V : c_{e,v} \in W \text{ for some } e \in E\}$ forms a vertex cover, and hence its size is at least $\beta \cdot n$. Note that each voter in V' disapproves at least one candidate in W, and hence at most $|V \setminus V'| \leq (1-\beta) \cdot n$ voters approve all candidates in W, which fails the second condition of the verification procedure.

□

8 Experiments: Participatory Budgeting in Amsterdam and Warsaw

In this section, we evaluate our voting rules on real-world participatory budgeting data collected by Faliszewski et al. [2023] from elections held in the Netherlands and Poland.

8.1 Participatory budgeting in Amsterdam with type budgets

We first focus on six district-level approval-based PB elections from Amsterdam that contain explicit type-budget constraints. Appendix D.1 provides a detailed discussion of the experimental results, together with tables summarizing the evaluation metrics. Here, we only briefly discuss the main conclusions of the experimental evaluation.

We show that both seqPhragmén and the greedy rule that repeatedly selects one project to maximize the PAV score (seqPAV) violate the specified type budgets in all six instances. MES also violates the specified type budgets in four of the six instances. By contrast, τ-seqPhragmén and τ-MES satisfy the type-budget constraints in all instances. Moreover, τ-MES achieves the highest welfare-per-cost ratio across all six instances. We also evaluate two variants of τ-MES. The first is τ-MES (Completion), a completion heuristic that first runs τ-MES until it terminates and then continues with τ-seqPhragmén on the residual instance.[5] The second is τ-MES (Add1U), which repeatedly increases voters' initial budgets and reruns τ-MES while the resulting outcome remains feasible, followed by a utilitarian greedy completion over the remaining feasible projects [Faliszewski et al., 2023]. τ-MES (Add1U) generally obtains higher total welfare, but at lower

[5]This approach is common in standard PB. We refer the reader to a detailed discussion by Peters and Skowron [2023].

efficiency. τ-MES (Completion) is an intermediate alternative, often matching τ-MES (Add1U) and occasionally improving on it slightly.

8.2 Participatory budgeting in Warsaw with overlapping types

We note that the Amsterdam PB instances have simple type constraint structures, where each project is subject to the global budget and exactly one category/neighborhood budget. We additionally perform experiments on PB elections in Warsaw from 2020 to 2024, where projects may have multiple types. For each election, we construct three variants using categories, beneficiaries, or both as types. For each of the 15 Warsaw instances, we sample 100 budget vectors by drawing every non-global type budget uniformly and independently from $[0, 2b_t]$, where b_t is the baseline budget equal to the total cost of the official winners of type t. We then evaluate the type-constrained version of seqPAV (τ-seqPAV), τ-seqPhragmén, τ-MES, τ-MES (Completion), and τ-MES (Add1U) on these sampled budget vectors.

Across the 1,500 sampled budget environments, τ-seqPhragmén, τ-MES, τ-MES (Completion), and τ-MES (Add1U) satisfy τ-EJR in every run, whereas τ-seqPAV does so in 76.9% of the runs. In terms of voter coverage, τ-MES (Completion) attains the lowest overall fraction of voters receiving no approved project, at 10.00%, closely followed by τ-seqPhragmén at 10.02% and τ-MES (Add1U) at 10.07%. In terms of total welfare, τ-MES (Add1U) performs best overall, although its advantage over τ-MES (Completion) is small and it is less computationally efficient [Peters and Skowron, 2023]. By contrast, τ-MES achieves by far the highest welfare-per-cost ratio. See Appendix D.2 for the full experimental details and tables summarizing all evaluation metrics.

9 Discussion

We introduced approval-based participatory budgeting with type constraints, where projects may belong to multiple types and each type is associated with its own budget limit. Since existing approval-based voting rules may fail to satisfy standard proportionality axioms even in the committee renewal special case, we introduce type-aware proportionality axioms. We propose type-aware adaptations of the Method of Equal Shares and Sequential Phragmén's rule, called τ-MES and τ-seqPhragmén, and prove positive guarantees for these rules under structured type constraints.

Several directions remain for future work. On the algorithmic side, it is natural to ask whether there exist polynomial-time rules that satisfy τ-EJR beyond multiwinner voting with nested type constraints, or that satisfy τ-PJR in τ-PB under natural non-laminar constraint structures. Another direction is to extend our model to include lower quotas on per-type spending. While our τ-PB framework captures type-aware participatory budgeting through upper spending limits, lower spending requirements are also common in practice and would be worthwhile to study. Our axioms, as defined herein, would require careful extension to meaningfully capture proportionality in this setting.

In multiwinner and PB settings without types, Brill and Peters [2023] introduced the PJR+ and EJR+ axioms, which strengthen PJR and EJR, respectively, and are verifiable in polynomial time. It would also be interesting to see whether an analogous modification can be made to our axioms. Such a modification would require more than a straightforward application of the approach used by Brill and Peters [2023] as it would need to account for each of the types an unselected candidate belongs to.

Approximate and randomized relaxations of our axioms are also natural directions for future work. In settings where exact proportionality is computationally intractable, it would be interesting to characterize how closely τ-PJR and τ-EJR can be approximated in polynomial time, and whether randomization can enable stronger proportionality guarantees.

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A Complexity of computing JR/EJR Outcomes for MW-Single

Theorem A.1. *Deciding whether an instance of* MW-Single *admits a feasible committee that satisfies JR is NP-complete, and $W[2]$-hard when parameterized by b.*

Proof. It can be verified in polynomial time whether a given committee satisfies JR [Aziz et al., 2017]. Hence, the problem is in NP. To prove hardness, we reduce from Set Cover. Let $(U, \mathcal{S}, s)$ be an instance of Set Cover, where $U = \{u_1, \ldots, u_p\}$ is the universe, $\mathcal{S} = \{S_1, \ldots, S_q\}$ is a family of subsets of U, and s is an integer; this is a yes-instance if U can be covered by at most s sets from $\mathcal{S}$.

We may assume that $\bigcup_{j=1}^{q} S_j = U$ and $1 \leq s \leq p$, since instances violating these conditions can be decided in polynomial time. Given such an instance, we construct an instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ of MW-Single as follows. For each element $u_i \in U$, create a voter $i \in N$. Thus, $n = p$ and $N = [n]$. Also, let $k = n$ and $b = s$. For each set $S_j \in \mathcal{S}$, create a candidate c_j. In addition, create a set D of k dummy candidates, so that $C = \{c_1, \ldots, c_q\} \cup D$. For each voter $i \in N$, define $A_i = \{c_j : u_i \in S_j\}$, that is, voter i approves exactly the candidates that correspond to sets containing u_i. Let the initial committee be $W_0 = D$.

Since $\frac{n}{k} = 1$, every voter with a non-empty approval set forms a 1-cohesive group. In our construction, no voter approves any dummy candidate. Therefore, a feasible committee W satisfies JR if and only if, for every voter $i \in N$, the committee W contains at least one candidate c_j such that $u_i \in S_j$.

Because W_0 consists entirely of dummy candidates, every non-dummy candidate included in W must replace one candidate from W_0. For W to be feasible, we must have $|W \setminus W_0| \leq b$. It follows that W satisfies JR if and only if the non-dummy candidates in W correspond to a set cover of U of size at most $b = s$.

Therefore, there exists a feasible committee satisfying JR if and only if the given Set Cover instance admits a set cover of size at most s. The construction is polynomial. Since Set Cover is NP-hard and $W[2]$-hard when parameterized by s, the same holds for our problem. □

Furthermore, even if W_0 already satisfies JR, it is hard to decide whether there exists a feasible committee that satisfies EJR.

Proposition A.1. *Given an instance of* MW-Single*, deciding whether a feasible EJR committee exists is NP-hard and is $W[2]$-hard when parameterized by b, even if the initial committee satisfies JR.*

Proof. We reduce from Set Cover. Let $(U, \mathcal{S}, s)$ be an instance of Set Cover, where $U = \{u_1, \ldots, u_p\}$ is the universe, $\mathcal{S} = \{S_1, \ldots, S_q\}$ is a family of subsets of U, and s is an integer.

As in the proof of Theorem A.1, assume that $\bigcup_{j=1}^{q} S_j = U$ and $1 \leq s \leq p$. Given such an instance, we construct an instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ of MW-Single as follows.

For each element $u_i \in U$, we create a block B_i of $q + 1$ voters. Thus, $n = p(q + 1)$. For each set $S_j \in \mathcal{S}$, we create a candidate c_j. For each element $u_i \in U$, we create a set A_i of q private candidates. Finally, we create a set D of p dummy candidates. Thus, $C = \{c_1, \ldots, c_q\} \cup \bigcup_{i \in [p]} A_i \cup D$. Set the committee size to $k = n = p(q + 1)$, and let $b = s$. Let the approval set of each voter $v \in B_i$ be $A_i \cup \{c_j : u_i \in S_j\}$. Let the initial committee be $W_0 = D \cup \bigcup_{i \in [p]} A_i$; note that $|W_0| = p + pq = k$. Note that each voter forms a 1-cohesive group; moreover, each voter $v \in B_i$ approves the candidates in $A_i \subset W_0$, so W_0 satisfies JR.

We will now argue that U has a cover of size at most s if and only if there exists a feasible committee satisfying EJR.

First, suppose that $\mathcal{S}' = \{S_j : j \in J\}$ is a set cover with $|J| \leq s$. Let D' be a subset of D of size $p - |J|$, and let

$$W = D' \cup \bigcup_{i \in [p]} A_i \cup \{c_j : j \in J\}.$$

Then $|W| = k$ and $|W \setminus W_0| = |J| \leq s = b$, so W is feasible.

We claim that W satisfies EJR. Indeed, every voter has q private candidates in W that she approves. Therefore, no group that deserves at most q candidates can witness a violation of EJR. Now, consider a group G that is $(q + 1)$-cohesive. Since $n = k$, we have $|G| \geq q + 1$. If $G = B_i$ for some $i \in [p]$, then $|G| = q + 1$ and the voters in B_i approve q candidates in $A_i \subset W$ as well as the candidate $c_j \in W$ such that S_j covers u_i, i.e., $q + 1$ candidates altogether. Hence, G cannot witness a violation of EJR. It remains to consider the case where G contains voters from two or more different

blocks. But then the set of candidates that are jointly approved by all voters in G is a subset of $\{c_1, \ldots, c_q\}$, so G is not $(q+1)$-cohesive, a contradiction.

Conversely, suppose that there exists a feasible committee W satisfying EJR, and let $\mathcal{S}' = \{S_j : c_j \in W\}$. We claim that $|\mathcal{S}'| \leq s$ and $\mathcal{S}'$ forms a cover of U. First, note that W_0 contains no candidates from $\{c_1, \ldots, c_q\}$ and hence W contains at most $b = s$ candidates from $\{c_1, \ldots, c_q\}$; hence, $|\mathcal{S}'| \leq s$. Next, fix an element $u_i \in U$; we will argue that u_i is covered by $\mathcal{S}'$. Indeed, since $|B_i| = q+1$, and $k = n$, the block B_i is $(q+1)$-cohesive and therefore some voter in B_i approves $q+1$ candidates in W. As $|A_i| = q$, this means that W contains a candidate c_j such that $u_i \in S_j$. Hence, $\mathcal{S}'$ contains a set S_j with $u_i \in S_j$.

Thus, the constructed instance of MW-Single admits a feasible EJR committee if and only if the original Set Cover instance has a cover of size at most s. The reduction is polynomial and $b = s$, so the problem is NP-hard and $W[2]$-hard when parameterized by b, even when the initial committee satisfies JR. □

Theorem A.1 implies that, unless $P = NP$, no polynomial-time computable voting rule conditionally satisfies JR. Indeed, given a polynomial-time rule that conditionally satisfies JR, we can go through all (polynomially many) committees in its output and check if they satisfy JR (which can be done in polynomial time, as argued by Aziz et al. [2017]): the input instance admits a JR committee if and only if all outputs of the rule satisfy JR.

Proposition 2. *In the special case of multiwinner voting, a group of voters is ℓ-deserving if and only if it is an ℓ-cohesive group in the sense of Definition 1.*

Proof. Consider an ℓ-cohesive group S in the sense of Definition 1; it satisfies (1) $|S| \geq \ell \cdot \frac{n}{k}$ and (2) $\left|\bigcap_{i \in S} A_i\right| \geq \ell$. By (2), $\cap_{i \in S} A_i$ contains a subset of size ℓ; let X be some such subset. Since each candidate has a unit cost and $T_1 = C$, we have $\text{cost}(X \cap T_1) = |X| = \ell$. On the other hand, (1) can be rewritten as $\ell \leq k \cdot \frac{|S|}{n}$. Since $b_1 = k$, we obtain $\text{cost}(X \cap T_1) \leq b_1 \cdot \frac{|S|}{n}$, i.e., S is ℓ-deserving.

Conversely, if S is ℓ-deserving, as witnessed by a set X of size at least ℓ, we have $\left|\bigcap_{i \in S} A_i\right| \geq \ell$. Further, for unit costs, substituting $T_1 = C$, $b_1 = k$ into the condition $\text{cost}(X \cap T_t) \leq b_t \cdot \frac{|S|}{n}$ translates into $\ell \leq |X| \leq k \cdot \frac{|S|}{n}$, or, equivalently, $|S| \geq \ell \cdot \frac{n}{k}$. Hence, S is ℓ-cohesive.

Thus, the two notions coincide, and the claim follows. □

B Comparison of Notions of EJR for Settings with Constraints

In this section, we compare the τ-EJR axiom introduced in our work with Restrained EJR by Mavrov et al. [2023] and with two axioms proposed by Masařík et al. [2024]: Base EJR and EJR with Feasibility Constraints.

To state the axioms from prior work, we write $E = (C, N, \mathcal{A}, \mathcal{P})$ to denote an election with feasibility constraints, where N is the set of voters, C is the set of candidates, $\mathcal{A} = (A_i)_{i \in N}$ is the approval profile (where $A_i \subseteq C$ for all $i \in N$), and $\mathcal{P} \subseteq 2^C$ is the family of feasible outcomes. The voters are assumed to have approval utilities, i.e., $u_i(T) = |A_i \cap T|$ for all $i \in N$, $T \subseteq C$.

Restrained Extended Justified Representation (R-EJR). Mavrov et al. [2023] define Restrained Extended Justified Representation for multiwinner elections with allocation constraints; we refer to this notion as R-EJR. In their model each feasible committee in $\mathcal{P}$ is of size at most k. A committee $\widehat{W}$ is *q-completable* if there exists a set W'' with $|W''| \leq q$ such that $\widehat{W} \cup W'' \in \mathcal{P}$. For a set of voters $S \subseteq N$ and a set of candidates $T \subseteq C$, let $A_S(T) = (\bigcap_{i \in S} A_i) \cap T$. The *endowment* of a set of voters S is $k_S = \lfloor |S| \cdot \frac{k}{n} \rfloor$. We say that S forms a *constraint-feasible blocking coalition* for a committee W if for every k_S-completable set $\widehat{W} \subseteq W$ with $|\widehat{W}| \leq k - k_S$ there exists a set W' with $|W'| \leq k_S$ such that $\widehat{W} \cup W' \in \mathcal{P}$ and $|A_S(\widehat{W} \cup W')| \geq \max_{i \in S} u_i(W) + 1$. Then, a committee $W \in \mathcal{P}$ satisfies R-EJR if there is no constraint-feasible blocking coalition $S \subseteq N$.

Base Extended Justified Representation (Base EJR). Masařík et al. [2024] introduce Base Extended Justified Representation (Base EJR). A group $S \subseteq N$ deserves ℓ candidates under Base EJR if, for every feasible set $T \in \mathcal{P}$, either (1) there exists a set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq \ell$ such that $T \cup X \in \mathcal{P}$, or (2) $\frac{|S|}{n} > \frac{\ell}{|T|+\ell}$[6]. A feasible outcome $W \in \mathcal{P}$ satisfies Base EJR if for every integer ℓ it holds that every group S that deserves ℓ candidates contains a voter $i \in S$ with $u_i(W) \geq \ell$.

Extended Justified Representation with Feasibility Constraints (EJR-FC). Masařík et al. [2024] also propose a stronger EJR notion, which they call Extended Justified Representation with Feasibility Constraints (EJR-FC). Given an outcome $W \in \mathcal{P}$, a group $S \subseteq N$ deserves ℓ candidates in W if, for every set $T \subseteq W$, either there exists a set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq \ell$ such that $T \cup X \in \mathcal{P}$, or $\frac{|S|}{n} > \frac{\ell}{|T|+\ell}$. A feasible outcome W satisfies EJR-FC if for every integer ℓ it holds that every group S that deserves ℓ candidates in W contains a voter $i \in S$ with $u_i(W) \geq \ell$. One can check that EJR-FC implies both Base EJR and R-EJR.

We first present examples in which our type-aware axioms impose stronger representation requirements. In the following examples, EJR-FC, and hence also Base EJR and R-EJR, permits outcomes that violate τ-EJR. These examples illustrate the additional requirements that arise from assigning each group a proportional share of every type budget.

Proposition B.1. *Consider the instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ of* MW-Single *defined as follows. Let $N = \{1, 2, 3, 4\}$ and let $C = \{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}$. Set $k = 4$, $b = 2$, and $W_0 = \{x_1, y_1, z_1, z_2\}$. The approval sets are $A_1 = A_2 = \{x_1, x_2\}$ and $A_3 = A_4 = \{y_1, y_2\}$. The candidates z_1, z_2, z_3, z_4 receive no approvals. Let $W = \{x_1, y_1, z_3, z_4\}$, $W^\star = \{x_1, x_2, y_1, y_2\}$. Then W fails EJR and τ-EJR, but satisfies EJR-FC, while $W^\star$ satisfies each of EJR, τ-EJR and EJR-FC.*

Proof. Intuitively, W uses the replacement budget on dummy candidates approved by no voter, despite the existence of feasible alternatives that provide additional representation to the two cohesive groups; we show that this interferes with τ-EJR, but not with EJR-FC.

We first observe that W violates (standard) EJR as well as τ-EJR. To this end, we will argue that the set of voters $Q = \{1, 2\}$ is 2-cohesive as well as 2-deserving; however, each voter in Q only approves one candidate in W. To see that Q is 2-cohesive, note that $|Q| = 2 \cdot \frac{n}{k}$ and $\bigcap_{i \in Q} A_i = \{x_1, x_2\}$. To see that it is 2-deserving, note that $T_1 = C$, $T_2 = C \setminus W_0$ with budgets k and b respectively. For $X = \{x_1, x_2\}$ we have $|X| = 2$, $X \subseteq \bigcap_{i \in Q} A_i$, $|X \cap T_1| = 2 = k \cdot \frac{|Q|}{n}$, and $|X \cap T_2| = 1 = b \cdot \frac{|Q|}{n}$.

In contrast, $W^\star$ satisfies EJR and τ-EJR: Q and $N \setminus Q$ are the only groups that are 2-cohesive (as well as 2-deserving), and each voter approves two candidates in $W^\star$.

Next, we show that W satisfies EJR-FC. Note first that if a group $S \subseteq N$ deserves $\ell > 0$ candidates in W then $|\bigcap_{i \in S} A_i| \geq \ell$. Indeed, for $T = \emptyset$ the condition $\frac{|S|}{n} > \frac{\ell}{|T|+\ell}$ is violated for all positive ℓ, so for S to deserve ℓ candidates there must exist a feasible set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq \ell$; this is only possible if $|\bigcap_{i \in S} A_i| \geq \ell$.

Now, consider a set S that deserves $\ell > 0$ candidates; as argued above, this implies $\cap_{i \in S} A_i \neq \emptyset$. Then $S \subseteq \{1, 2\}$ or $S \subseteq \{3, 4\}$. By symmetry, it suffices to consider two cases: (1) $S = \{1\}$ and (2) $S = \{1, 2\}$. We will argue that in each of these cases S deserves at most one candidate. Indeed, let $T = \{z_3, z_4\}$, and consider $X \subseteq \bigcap_{i \in S} A_i \subseteq \{x_1, x_2\}$. If $X = \{x_1, x_2\}$, then $T \cup X$ is not feasible, since $z_3, z_4 \notin W_0$ and $b = 2$. On the other hand, the condition $\frac{|S|}{n} > \frac{\ell}{|T|+\ell}$ becomes $4\ell < \ell + 2$ (if $S = \{1\}$) or $2\ell < \ell + 2$ (if $S = \{1, 2\}$); in either case, this condition is not satisfied by $\ell \geq 2$. Thus, no group of voters deserves 2 candidates. As each voter approves one candidate in W, it follows that W satisfies EJR-FC (and hence R-EJR and Base EJR).

Note that $W^\star$ satisfies EJR-FC as well. Indeed, since each voter approves two candidates, no group deserves ℓ candidates for $\ell > 2$, and each voter approves two candidates in $W^\star$. □

[6]For sets T with $|T| \neq 0$, this inequality can be equivalently stated as $\frac{|S|}{\ell} > \frac{n-|S|}{|T|}$. Since only one of X and T can be included, this inequality compares the proportional claims of S to get ℓ representatives and the rest of the voters to get $|T|$ representatives.

Our next example concerns an election with two disjoint types, and can be interpreted as running two independent elections in parallel.

Proposition B.2. *Consider the following instance of multiwinner voting with type constraints:* $n = 6$, $C = A \cup B$, *where* $A = \{x_1, x_2, x_3\}$, $B = \{y_1, y_2, y_3, y_4\}$, $\mathcal{H} = (T_1, T_2)$, *where* $T_1 = A$, $T_2 = B$, $b_1 = 2$ *and* $b_2 = 4$, $A_i = \{x_1, y_1, y_2\}$ *for* $i = 1, 2, 3$ *and* $A_i = \{x_2, x_3\}$ *for* $i = 4, 5, 6$. *Then the committee* $W = C \setminus \{x_1\}$ *satisfies EJR-FC, but not* τ*-EJR.*

Proof. We first argue that W fails τ-EJR. To see this, let $S = \{1, 2, 3\}$. Intuitively, since S constitutes half of the voters and every voter in S has identical preferences, they should be entitled to decide on half of the candidates of type T_1, and half of the candidates of type T_2. Formally, we have $n = 2 \cdot |S|$ and hence S is 3-deserving, as witnessed by $\{x_1, y_1, y_2\}$: indeed, $\text{cost}(X \cap T_1) = 1 = b_1 \cdot \frac{|S|}{n}$, $\text{cost}(X \cap T_2) = 2 = b_2 \cdot \frac{|S|}{n}$. However, we have $|W \cap \bigcup_{i \in S} A_i| = 2$, so W fails τ-EJR.

However, W satisfies EJR-FC, and thus R-EJR. Note that voters in $N \setminus S$ approve only x_2 and x_3, both of which belong to W. Hence, no subset of $N \setminus S$ can witness a violation of EJR-FC.

Thus, it suffices to consider subsets of S; moreover, as W contains two candidates approved by voters in S, we only need to argue that S does not deserve three candidates. Consider $T = \{x_2, x_3\}$; note that $T \subseteq W$. There is no set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq 3$ such that $T \cup X$ is feasible: indeed, the only size-3 subset of $\bigcap_{i \in S} A_i$ is $X = \{x_1, y_1, y_2\}$, and $|(T \cup X) \cap T_1| = 3$. Moreover, it is not the case that $\frac{|S|}{n} = \frac{1}{2} > \frac{3}{|T|+3} = \frac{3}{5}$. Thus, T witnesses that S does not deserve 3 candidates. Consequently, W satisfies EJR-FC. □

Conversely, there are also instances in which τ-EJR permits an outcome that is ruled out by R-EJR and Base EJR.

Proposition B.3. *Consider the instance* $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ *of* MW-Single *defined as follows. Let* $N = \{1, 2, 3\}$ *and let* $C = \{a, c, d_1, d_2\}$. *Set* $k = 2$, $b = 1$, *and* $W_0 = \{d_1, d_2\}$. *The approval sets are* $A_1 = A_2 = \{a\}$ *and* $A_3 = \{c\}$. *Then the feasible outcome* $W = W_0$ *satisfies* τ*-EJR, but fails Base EJR and R-EJR.*

Proof. To argue that W satisfies τ-EJR, we will show that no group of voters is ℓ-deserving for $\ell > 0$. It suffices to consider $S = \{1, 2\}$: we have $b \cdot \frac{|S|}{n} = \frac{2}{3} < 1$.

By contrast, $S = \{1, 2\}$ deserves one candidate under Base EJR. Indeed, consider a feasible set T: either $T = \emptyset$, in which case $T \cup \{a\}$ is feasible, or $|T| \geq 1$, in which case $\frac{|S|}{n} = \frac{2}{3} > \frac{1}{|T|+1}$. As W provides zero utility to both voters in S, it violates Base EJR (and hence EJR-FC). It also violates R-EJR, since the group S, with $k_S = \lfloor |S| \cdot \frac{k}{n} \rfloor = 1$, forms a constraint-feasible blocking coalition: the 1-completable subsets $\widehat{W} \subseteq W$ satisfying $|\widehat{W}| \leq k - k_S = 1$ are precisely $\varnothing$, $\{d_1\}$, and $\{d_2\}$. For $W' = \{a\}$ it holds that $\widehat{W} \cup W'$ is feasible, and all voters in S prefer $\widehat{W} \cup W'$ to W. □

Together with Propositions B.1 and B.2, Proposition B.3 shows that our type-aware axioms and the axioms of Mavrov et al. [2023] and Masařík et al. [2024] are generally incomparable[7].

C Incomparability under Droop Quotas

The separation in Proposition B.3 relies on the Hare-style proportional share used in our definition of deservingness. In that example, a group consisting of two out of three voters cannot claim a candidate that consumes the single replacement slot, since its share of the replacement budget is only 2/3. One may therefore ask whether the comparison with the axioms of Mavrov et al. [2023] and Masařík et al. [2024] changes if our notion is strengthened using Droop-style quotas [Casey and Elkind, 2025]. We show that the incomparability persists.

[7]This incomparability is not solely due to our use of the Hare quota $\frac{n}{k}$. In Appendix C, we show that it persists under an analogous strengthening based on Droop quotas [Casey and Elkind, 2025], even in the MW-Single setting.

Definition C.1 (Droop-τ-EJR)**.** Consider an instance $\mathcal{I} = (N, C, (A_i)_{i\in N}, W_0, k, b)$ of MW-Single. A group $S \subseteq N$ is *Droop-τ-deserving* of ℓ candidates if there exists a set $X \subseteq \bigcap_{i\in S} A_i$ with $|X| \geq \ell$ such that $\frac{|S|}{n} > \frac{|X|}{k+1}$ and $\frac{|S|}{n} > \frac{|X\setminus W_0|}{b+1}$. A feasible outcome W satisfies Droop-τ-EJR if, for every $\ell \geq 1$ and every group S that is Droop-τ-deserving of ℓ candidates, there exists a voter $i \in S$ with $u_i(W) \geq \ell$.

Proposition C.1. *Droop-τ-EJR does not imply Base EJR, even in the* MW-Single *setting.*

Proof. Let $N = \{1, 2\}$, let $W_0 = \{d_1, d_2\}$, and set $k = 2$ and $b = 1$. Let a_1, a_2, a_3, a_4 be candidates outside W_0, and let $A_1 = \{a_1, a_2, a_3, a_4\}$ and $A_2 = \emptyset$. Consider the feasible outcome $W = W_0$.

We first show that W satisfies Droop-τ-EJR. The only group with a nonempty common approval set is $S = \{1\}$. Every nonempty witness $X \subseteq A_1$ contains at least one candidate outside W_0, so Droop-τ-deservingness would require $\frac{|S|}{n} = \frac{1}{2} > \frac{|X\setminus W_0|}{b+1} \geq \frac{1}{2}$, which is impossible. Thus, no group has a positive claim under Droop-τ-EJR.

By contrast, $S = \{1\}$ deserves one candidate under Base EJR. Fix any feasible set T. If T contains some a_j, take $X_T = \{a_j\}$, so $T \cup X_T = T$ is feasible. If T contains no newcomer and $|T| \leq 1$, then adding any a_j remains feasible. The only remaining case is $T = W_0$, for which $\frac{|S|}{n} = \frac{1}{2} > \frac{1}{|T|+1} = \frac{1}{3}$. Hence, S deserves one candidate under Base EJR. Since voter 1 obtains utility zero from W, the outcome violates Base EJR. □

The separation arises because Base EJR may use a different augmentation X_T for different feasible sets T, whereas Droop-τ-deservingness requires a single witness set to satisfy all proportional budget requirements.

Proposition C.2. *Droop-τ-EJR does not imply R-EJR, even in the* MW-Single *setting.*

Proof. Let $N = \{1, 2\}$, let $W_0 = \{d_1, d_2\}$, and set $k = 2$ and $b = 1$. Let $a, c \notin W_0$, with $A_1 = \{a\}$ and $A_2 = \{c\}$. Again consider $W = W_0$.

Neither singleton is Droop-τ-deserving of one candidate, since for either singleton the replacement-budget condition would require $\frac{1}{2} > \frac{1}{b+1} = \frac{1}{2}$. The grand coalition has no commonly approved candidate. Hence, W satisfies Droop-τ-EJR.

However, W violates R-EJR. Consider $S = \{1\}$. Its proportional committee allowance is $k_S = \lfloor |S|k/n \rfloor = 1$. For every relevant retained set $\widehat{W} \subseteq W$ with $|\widehat{W}| \leq k - k_S = 1$, the set $\widehat{W} \cup \{a\}$ is feasible and gives voter 1 utility one rather than zero. Thus, S forms a blocking coalition under R-EJR. □

Since EJR-FC implies R-EJR, this example also shows that Droop-τ-EJR does not imply EJR-FC.

These examples show that replacing Hare shares with the stronger Droop-style thresholds does not eliminate the conceptual distinction between our approach and the existing axioms. Even in committee renewal, Droop-τ-EJR neither implies Base EJR nor R-EJR. Thus, the incomparability identified in the main text reflects differences in how group entitlements are defined, rather than merely the choice between Hare and Droop quotas.

D Experimental Results

D.1 Experiments on PB Instances in Amsterdam

We focus on six district-level approval-based PB elections from Amsterdam that contain explicit type-budget constraints: Kleine Wereld (KW), Bloemenbuurt (BB), Centrum Begroot (CB), Geuzenveld Slotermeer (GS), Oud Oost (OO), and IJburg Zeeburgereiland (IZ).

The six instances vary considerably in scale, ranging from BB, with 219 voters and 24 projects, to CB, with 8,520 voters and 85 projects. Their overall budgets lie between €250,000 and €500,000. In five of these instances, category budgets are specified alongside the overall budget, with each category budget imposing an upper bound on the total cost of selected projects in the corresponding category. In Centrum Begroot, by contrast, the data specifies neighborhood budgets, each imposing an

Instance	Voters	Projects	Budget	#types	Min type budget	Max type budget	Avg. type budget
KW	426	52	€250,000	6	€33,000	€54,000	€41,667
BB	219	24	€250,000	3	€53,000	€115,000	€83,333
CB	8,520	85	€500,000	3	€100,000	€250,000	€166,667
GS	5,510	97	€400,000	3	€100,000	€200,000	€133,333
OO	2,324	24	€300,000	4	€57,312	€90,610	€75,000
IZ	2,037	25	€300,000	4	€56,916	€91,354	€75,000

Table 2: Basic statistics of the six Amsterdam instances used in the experiments.

Instance	Type	Budget
KW	Armoede	€52,000
KW	Eenzaamheid	€37,000
KW	Groenonderhoud straten & pleinen	€35,000
KW	Jeugdactiviteiten	€54,000
KW	Rattenpreventie	€39,000
KW	Sportactiviteiten	€33,000
BB	Jeugd	€82,000
BB	Groen	€53,000
BB	Ontmoeting	€115,000
CB	Oostelijke Eilanden	€150,000
CB	Haarlemmerhouttuinen	€100,000
CB	Other	€250,000
GS	Straten pleinen en parken	€200,000
GS	Gezondheid cultuur en kansen voor iedereen	€100,000
GS	Samen dingen doen	€100,000
OO	Meer groen in de buurt	€90,610
OO	Kindvriendelijke buurt	€57,312
OO	Kinder- en jongerenactiviteiten	€71,364
OO	Klimaat en duurzaamheid	€80,715
IZ	Meer groen in de buurt	€91,354
IZ	Plekken voor jongeren	€71,896
IZ	Kinder- en jongerenactiviteiten	€56,916
IZ	Minder zwerfvuil/grof afval	€79,834

Table 3: Type budgets in the six Amsterdam instances.

upper bound on the total cost of selected projects in the corresponding neighborhood. Specifically, KW contains 426 voters, 52 projects, and six category budgets under a total budget of €250,000. BB contains 219 voters, 24 projects, and three category budgets under €250,000. CB contains 8,520 voters, 85 projects, and three neighborhood budgets under €500,000. GS contains 5,510 voters, 97 projects, and three category budgets under €400,000. OO contains 2,324 voters, 24 projects, and four category budgets under €300,000. IZ contains 2,037 voters, 25 projects, and four category budgets under €300,000. Table 2 reports their basic statistics, while Table 3 lists the type budgets in each instance.

Existing rules using the overall budget only. For each instance, we first ignore the type budgets and run three unconstrained approval-based rules using the overall budget only:

- **MES**, the standard Method of Equal Shares **without** type constraints
- **seqPhragmén**, the standard seqPhragmén voting rule **without** type constraints
- **seqPAV**, the greedy rule that repeatedly selects one project to maximize the Proportional Approval Voting (PAV) score

Instance	#Type Constraints*	seqPAV	seqPhragmén	MES	τ-seqPhragmén	τ-MES
KW	6	✗	✗	✔	✔	✔
BB	3	✗	✗	✗	✔	✔
CB	3	✗	✗	✗	✔	✔
GS	3	✗	✗	✔	✔	✔
OO	4	✗	✗	✗	✔	✔
IZ	4	✗	✗	✗	✔	✔

Table 4: Whether outcomes selected by the rules satisfy the specified type budgets. * The overall budget constraint is not counted.

Instance	Total Welfare ↑					Welfare / €1,000 ↑				
	τ-seqPAV	τ-seqPhragmén	τ-MES	τ-MES (Completion)	τ-MES (Add1U)	τ-seqPAV	τ-seqPhragmén	τ-MES	τ-MES (Completion)	τ-MES (Add1U)
KW	3,324	3,729	2,355	3,729	**3,749**	13.9	19.0	**39.4**	19.0	18.9
BB	1,397	1,777	1,572	1,777	**1,802**	8.4	11.8	**14.6**	11.8	11.5
CB	60,928	88,413	68,759	**88,943**	88,413	126.2	186.0	**322.0**	187.5	186.0
GS	10,253	13,618	5,888	13,618	**13,666**	25.9	34.4	**69.6**	34.4	34.8
OO	**22,538**	21,697	17,859	21,697	21,697	88.2	95.5	**109.7**	95.5	95.5
IZ	7,680	**9,088**	4,728	8,670	8,670	28.8	36.6	**51.5**	37.2	37.2

Table 5: τ-seqPAV, τ-seqPhragmén, τ-MES, τ-MES (Completion), and τ-MES (Add1U) on all six Amsterdam instances. Within each metric block, the numerically largest entry in each row is highlighted in bold.

We then check whether the selected outcome violates the specified category or neighborhood budgets. Table 4 shows that both seqPhragmén and seqPAV violate the specified budgets in all six instances. MES also violates the specified budgets in four of the six cases. By contrast, τ-seqPhragmén and τ-MES satisfy the type-budget constraints in all instances.

Comparison under type constraints. Having established that unconstrained rules may violate the specified type budgets, we next solve all six Amsterdam instances under explicit type-budget constraints. For each instance, we compare five voting rules:

1. **τ-seqPAV**, which greedily adds a feasible project maximizing the increase in PAV score
2. **τ-seqPhragmén**, seqPhragmén with type constraints
3. **τ-MES**, Algorithm 1
4. **τ-MES (Completion)**, a completion heuristic that first runs τ-MES until it terminates and then continues with τ-seqPhragmén on the residual instance [8]
5. **τ-MES (Add1U)**, an adaptation of τ-MES that repeatedly increases voters' initial budgets and reruns τ-MES while the resulting outcome remains feasible, followed by a utilitarian greedy completion over the remaining feasible projects [Faliszewski et al., 2023].

In Table 5, we report two evaluation metrics. The first is total welfare, $\sum_{i \in N} u_i(W)$, and the second is the welfare-per-cost ratio, $\frac{\sum_{i \in N} u_i(W)}{\text{cost}(W)}$, which measures the approval utility obtained per unit of budget spent. Across all six instances, τ-MES achieves the highest welfare-per-cost ratio. τ-MES (Add1U) generally obtains higher total welfare, but at lower efficiency. τ-MES (Completion) is an intermediate alternative, often matching τ-MES (Add1U) and occasionally improving on it slightly.

[8] This approach is common in standard PB. We refer the reader to a detailed discussion by Peters and Skowron [2023].

Year	#Projects	#Category labels	#Beneficiary labels	Multi-category projects	Multi-beneficiary projects	Avg. category labels/project	Avg. beneficiary labels/project
2020	101	9	6	72	85	2.13	3.50
2021	106	10	7	92	101	2.33	3.93
2022	129	10	8	97	114	2.20	3.60
2023	138	10	7	112	130	2.35	4.14
2024	118	9	3	99	83	2.79	1.70

Table 6: Label statistics for the five Warsaw datasets.

Instance	τ-seqPAV	τ-seqPhragmén	τ-MES	τ-MES (Completion)	τ-MES (Add1U)
W20-cat	7.02	6.39	13.95	6.38	**6.37**
W20-ben	9.07	7.59	14.64	**7.57**	7.61
W20-comb	11.93	11.07	26.19	**10.99**	11.07
W21-cat	7.45	6.42	14.56	**6.40**	6.43
W21-ben	10.49	**8.47**	17.31	**8.47**	8.51
W21-comb	14.17	**12.18**	27.34	12.19	12.34
W22-cat	11.16	9.43	23.53	**9.39**	9.62
W22-ben	15.50	12.05	22.64	**12.02**	12.06
W22-comb	19.96	**17.01**	38.56	**17.01**	17.08
W23-cat	10.30	9.21	29.21	**9.12**	9.34
W23-ben	18.21	**12.59**	21.74	12.64	12.79
W23-comb	19.97	16.18	38.96	**16.10**	16.20
W24-cat	7.04	5.94	15.40	5.94	**5.85**
W24-ben	10.38	**5.45**	10.78	5.46	5.50
W24-comb	12.77	10.34	27.37	**10.26**	10.29
Overall	12.36	10.02	22.81	**10.00**	10.07

Table 7: Average percentage of voters receiving no approved project on PB instances in Warsaw. Lower values are better, and the smallest entry in each row is highlighted in bold. The overall row averages the percentage over all 1,500 runs.

D.2 Experiments on PB Instances in Warsaw

We next consider 15 PB instances derived from five citywide approval-based elections in Warsaw, one for each year from 2020 to 2024. From each yearly dataset, we construct three variants: a category-based instance, a beneficiary-based instance, and a combined instance using both categories and beneficiaries as types.

The resulting type systems are substantially richer than in Amsterdam. The 2020 dataset contains 9 category labels and 6 beneficiary labels, yielding 15 non-global types in the combined variant. The corresponding numbers are 10 and 7 in 2021, 10 and 8 in 2022, 10 and 7 in 2023, and 9 and 3 in 2024. Multi-label projects are common throughout: for example, 72 of the 101 projects in 2020 and 112 of the 138 projects in 2023 belong to multiple categories, while 85 of the 101 projects in 2020 and 130 of the 138 projects in 2023 belong to multiple beneficiary groups. Table 6 summarizes these label statistics.

Since these datasets do not specify type budgets, we calibrate a baseline budget b_t for each non-global type which is equal to the total cost of the official winners of type t. We then sample each non-global type budget from $[0, 2b_t]$ and run the rules.

Overlapping types in Warsaw. For each of the 15 Warsaw instances, we sample 100 budget vectors by drawing every non-global type budget uniformly and independently from $[0, 2b_t]$, where b_t is the baseline budget equal to the total cost of the official winners of type t. For each instance, we evaluate five rules: τ-seqPAV, τ-seqPhragmén, τ-MES, τ-MES (Completion), and τ-MES (Add1U). Tables 7 to 9 report three statistics: the average fraction of voters receiving no approved project,

Instance	τ-seqPAV	τ-seqPhragmén	τ-MES	τ-MES (Completion)	τ-MES (Add1U)
W20-cat	348.6	377.8	233.6	378.3	**378.9**
W20-ben	314.7	358.4	250.2	**359.7**	359.0
W20-comb	257.5	280.2	143.3	**281.1**	280.8
W21-cat	348.9	400.3	241.8	402.0	**402.9**
W21-ben	277.4	348.8	224.9	**349.3**	349.2
W21-comb	229.4	269.7	147.6	**270.5**	269.5
W22-cat	238.3	275.0	156.7	275.7	**276.9**
W22-ben	205.0	273.2	166.3	**273.9**	273.0
W22-comb	166.5	199.3	94.5	**199.7**	198.9
W23-cat	225.7	250.4	104.9	**251.7**	251.1
W23-ben	166.8	248.5	174.5	**248.9**	248.3
W23-comb	148.7	181.7	79.0	182.6	**184.7**
W24-cat	268.0	297.8	176.2	299.1	**300.8**
W24-ben	235.6	318.7	243.5	321.6	**322.8**
W24-comb	205.9	232.9	118.6	235.3	**235.9**
Overall	242.5	287.5	170.4	288.6	**288.8**

Table 8: Average welfare, in thousands, on PB instances in Warsaw. Within each row, the best entry is highlighted in bold.

average total welfare, and average welfare per 1,000 PLN. We also check whether the outcomes satisfy τ-EJR. Across the 1,500 sampled budget environments, τ-seqPhragmén, τ-MES, τ-MES (Completion), and τ-MES (Add1U) satisfy τ-EJR in every run, whereas τ-seqPAV does so in 76.9% of the runs. In terms of voter coverage, τ-MES (Completion) attains the lowest overall fraction of voters receiving no approved project, at 10.00%, closely followed by τ-seqPhragmén at 10.02% and τ-MES (Add1U) at 10.07%. In terms of total welfare, τ-MES (Add1U) performs best overall, although its advantage over τ-MES (Completion) is small and it is less computationally efficient [Peters and Skowron, 2023]. By contrast, τ-MES achieves by far the highest welfare-per-cost ratio. Although Section 4 shows that the compared rules do not all guarantee τ-EJR on arbitrary instances with type constraints, these results are consistent with empirical evidence that proportionality axioms are often satisfied in practice [Boehmer et al., 2026].

Instance	τ-seqPAV	τ-seqPhragmén	τ-MES	τ-MES (Completion)	τ-MES (Add1U)
W20-cat	18.9	20.5	**37.3**	20.5	20.6
W20-ben	22.6	26.7	**54.9**	26.7	27.0
W20-comb	26.5	30.5	**83.8**	30.5	31.4
W21-cat	19.5	23.3	**48.4**	23.3	24.2
W21-ben	21.2	28.9	**55.6**	28.8	29.2
W21-comb	27.6	35.5	**109.1**	35.7	37.2
W22-cat	11.2	13.0	**36.9**	13.1	13.3
W22-ben	8.9	12.7	**36.0**	12.8	12.9
W22-comb	12.7	16.4	**59.0**	16.4	17.2
W23-cat	9.6	11.4	**30.7**	11.4	11.8
W23-ben	10.9	16.2	**40.2**	16.1	16.2
W23-comb	14.7	18.5	**73.1**	18.8	19.5
W24-cat	11.3	12.9	**27.1**	12.9	13.1
W24-ben	9.0	12.3	**21.2**	12.4	12.5
W24-comb	15.0	17.8	**38.7**	17.9	18.5
Overall	16.0	19.8	**50.1**	19.8	20.3

Table 9: Average welfare per 1,000 PLN on PB instances in Warsaw. Project costs are measured in PLN. Within each row, the best entry is highlighted in bold.