Proportionality in Participatory Budgeting with Type Constraints

Xingchen Sha[*]
Department of Computer Science
Northwestern University
`xingchen.sha@u.northwestern.edu`

Jeremy Vollen[*]
Department of Computer Science
Northwestern University
`vollen@northwestern.edu`

Matthew M. Casey
Department of Computer Science
Northwestern University

Edith Elkind
Department of Computer Science
Northwestern University

Abstract

We study proportional representation in approval-based participatory budgeting with type constraints, where each project belongs to one or more types, and each type is associated with a budget constraint. An important special case of our model is committee renewal, where we are given a committee of size k, and can replace up to $b \leq k$ committee members. We show that, even in this simple setting, classic approval-based voting rules may fail to output outcomes that satisfy standard proportionality axioms. To tackle this issue, we formulate type-aware proportionality axioms—τ-PJR/EJR/FJR—which define a group deserving representation in terms of the group's ability to agree on a set of projects that fit within the group's proportional share of every type budget. While these axioms are not satisfied by existing voting rules, we put forward type-aware adaptations of the Method of Equal Shares and the Phragmén's rule, which we term τ-MES and τ-Phragmén. We prove that τ-MES satisfies τ-PJR for laminar types; for multiwinner voting with nested types, τ-MES satisfies τ-EJR, whereas τ-Phragmén satisfies τ-PJR. These results are essentially tight: the respective proportionality guarantees fail once the structural assumptions are relaxed. Moreover, we show that no polynomial-time rule can satisfy τ-EJR for laminar types. Thus, our proposed rule achieves the strongest guarantee one can hope for in polynomial time in this setting. For arbitrary types, we obtain a powerful existence result, but it comes with computational limitations: our strongest axiom—τ-FJR—is always satisfiable, but computing a feasible outcome that satisfies even the weaker τ-PJR axiom is strongly NP-hard. We complement our theoretical analysis with experiments on real-world participatory budgeting datasets, using elections in Amsterdam with explicit category/neighborhood budgets and elections in Warsaw with synthetically generated type budgets.

1 Introduction

Participatory Budgeting (PB) has become an important mechanism for enabling citizens to participate directly in public spending decisions, and has been widely adopted by municipalities around the world, including in Brazil, the United States, France, Poland, the Netherlands, Spain, and South Africa [Cabannes, 2004, Sintomer et al., 2008, Blakey, 2008, Röcke, 2014, Masiya et al., 2021]. In a typical PB process, voters express support for projects, and the planner selects a subset of projects

---

[*] Equal contribution.

Preprint.

whose cost does not exceed a given budget. A key concern in this setting is proportionality, i.e., ensuring that groups of voters with common interests are allocated a fair share of the total budget [Peters et al., 2021, Brill et al., 2023, Mavrov et al., 2023, Kraiczy and Elkind, 2023, Aziz and Lee, 2021]; see also the surveys by Aziz and Shah [2020] and Rey et al. [2025b].

However, in many practical PB scenarios, a single budget constraint does not fully capture the structure of feasible outcomes. Indeed, modern public agencies often allocate separate budgets across different programs, policy areas, constituencies, or target populations [Wildavsky, 1964, Rubin, 2019, Schick, 2008]. A city may also wish to limit how much can be spent on transportation, education, or public safety projects. More broadly, some funds may be earmarked for specific purposes and therefore may only be used to support projects of a certain type [Ebdon and Franklin, 2006]. The perspective of type-based constraints is also relevant in a range of domains besides PB, such as machine learning and data mining [Cui et al., 2024]. For example, in clustering under fairness constraints, a type may represent a protected group or demographic category, and the budget of a type bounds how many selected centers, representatives, or assignments may be associated with that group [see, e.g., Abbasi et al., 2021, Chierichetti et al., 2017]. In participatory budgeting, an additional complication is that a given project may belong to multiple types—e.g., 'building a cycle path in Uptown' is both a 'transportation infrastructure project' and an 'Uptown project'—and hence may be subject to multiple type constraints. Thus, an important research challenge is to extend the mathematical framework of participatory budgeting so as to express constraints of this form, and, in particular, to formulate suitable notions of proportionality and fairness in the presence of per-type budgets.

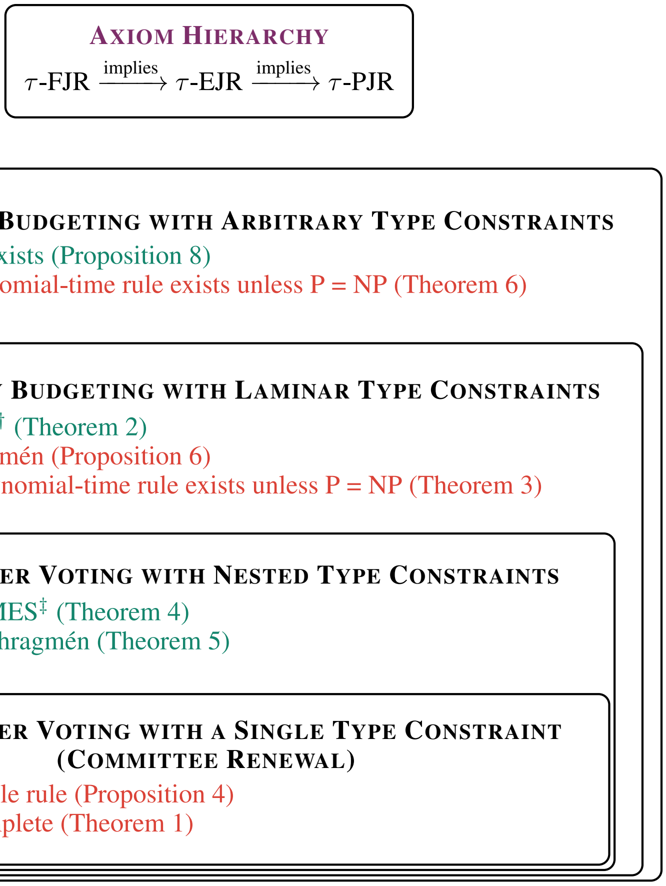
A number of recent papers consider participatory budgeting and/or multiwinner voting (a special case of participatory budgeting, where all projects have the same cost) with constraints, focusing primarily on maximizing the social welfare subject to constraints [Jain et al., 2021, Fluschnik et al., 2019, Bredereck et al., 2018, Hershkowitz et al., 2021, Kellerer et al., 2004, Goyal et al., 2023]. The literature on proportionality in PB with constraints is considerably more scarce, and purports to handle a very broad class of constraints [Mavrov et al., 2023, Masařík et al., 2024]; as a consequence, it puts forward complex definitions of proportionality that sometimes fail to rule out obviously undesirable outcomes (see Example 1 and Example 2 for an extended discussion). Moreover, the expressivity of these frameworks makes it challenging to design practical voting rules for them. It is therefore natural to ask whether, by focusing specifically on per-type budget constraints, we may be able to (1) formulate succinct and easy-to-parse axioms that are tailored to capture proportionality in this setting, and (2) propose appealing voting rules that satisfy these axioms.

Our Contribution We propose a family of proportionality axioms for *participatory budgeting with type constraints (τ-PB)*, as well as voting rules that satisfy these axioms.

In our model, each project belongs to one or more types, and each type is associated with a budget limit. The planner must choose a subset of projects so that the total spending on each type does not exceed the budget for that type. This model generalizes standard participatory budgeting (and hence multiwinner voting), while capturing a wide range of settings in which different classes of projects compete for distinct but interacting budget resources. This includes, e.g., committee selection with upper quotas, where each type represents a demographic group, region, or profession, and the budget of a type specifies the maximum number of selected candidates from that category [Bredereck et al., 2018, Celis et al., 2018].

Our starting point for defining proportionality is the family of justified representation (JR) axioms, which were first defined in the context of multiwinner voting [Aziz et al., 2017, Sánchez-Fernández et al., 2026] and then extended to participatory budgeting [Peters et al., 2021, Los et al., 2022]. As a first step, we consider the simplest variant of PB with type constraints — committee renewal — where we are given a size-k committee and are allowed to replace up to $b \leq k$ committee members. We show that, even in this simple case, classic multiwinner rules that offer strong guarantees in the 'vanilla' setting may fail to output outcomes that satisfy JR.

Motivated by these limitations, we propose type-aware analogs of common justified representation axioms, namely, τ-PJR, τ-EJR, and τ-FJR, as well as adaptations of two well-known PB voting rules with strong proportionality guarantees, namely, the Method of Equal Shares (MES) and Phragmén's rule. Our adapted rules, τ-MES and τ-Phragmén, are polynomial-time computable and provide proportionality guarantees in a number of important settings more general than committee renewal.



Axiom Hierarchy

τ-FJR $\xrightarrow{\text{implies}}$ τ-EJR $\xrightarrow{\text{implies}}$ τ-PJR

Participatory Budgeting with Arbitrary Type Constraints

✓ τ-FJR: always exists (Proposition 8)

✗ τ-PJR: No polynomial-time rule exists unless P = NP (Theorem 6)

Participatory Budgeting with Laminar Type Constraints

✓ τ-PJR: τ-MES† (Theorem 2)

✗ τ-PJR: τ-Phragmén (Proposition 6)

✗ τ-EJR: No polynomial-time rule exists unless P = NP (Theorem 3)

Multiwinner Voting with Nested Type Constraints

✓ τ-EJR: τ-MES‡ (Theorem 4)

✓ τ-PJR: τ-Phragmén (Theorem 5)

Multiwinner Voting with a Single Type Constraint (Committee Renewal)

✗ τ-PJR: Thiele rule (Proposition 4)

✗ JR: NP-complete (Theorem 1)

Figure 1: Summary of proportionality results for τ-PB and its special cases. Nested boxes indicate that each outer setting is more general, while each inner box is a special case. The superscript † indicates that the guarantee does not extend beyond Participatory Budgeting with Laminar Type Constraints (Proposition 5), and the superscript ‡ indicates that the guarantee does not extend beyond Multiwinner Voting with Nested Type Constraints (Proposition 7).

When the constraint structure forms a laminar set family, we show that τ-MES satisfies τ-PJR. For the more restrictive setting of multiwinner voting with a nested constraint structure, we prove that τ-MES satisfies the stronger τ-EJR axiom, whereas τ-Phragmén satisfies τ-PJR.

We also give a number of negative results showing that our algorithmic results do not hold (and, in some cases, cannot be improved unless P = NP) under stronger proportionality properties or more general constraint structures. In particular, we show that, unless P = NP, there is no polynomial-time algorithm which satisfies τ-EJR, even for disjoint constraint structures, a special case of laminar types. Moreover, we demonstrate our analysis of τ-MES is tight. For any non-laminar constraint family over a given set of candidates, τ-MES may fail to satisfy τ-PJR. We analogously show that our τ-EJR result for multiwinner voting with nested constraints no longer holds if we generalize from multiwinner voting to heterogeneous costs, or from nested to laminar constraints.

For arbitrary types, we can no longer compute proportional outcomes in polynomial time: computing a feasible outcome that satisfies even τ-PJR (our weakest axiom) is strongly NP-hard. However, we obtain a powerful existence result by showing that τ-FJR (our strongest axiom) is always satisfiable. Figure 1 summarizes these results.

We also evaluate our voting rules on real-world participatory budgeting datasets. Our experiments show that existing rules that ignore type constraints often produce infeasible outcomes. For the participatory budgeting elections in Warsaw from 2020 to 2024, our proposed rules satisfy τ-EJR in all 1,500 instances with sampled type budgets. In contrast, the greedy rule that maximizes an objective function over voter utilities violates τ-EJR in 23.1% of the runs.

2 Related Work

Proportionality in participatory budgeting The study of proportionality in participatory budgeting originates in the approval-based multiwinner voting literature, a special case of participatory budgeting. In that setting, the *justified representation* (JR) axiom and a number of related stronger axioms have been introduced [Aziz et al., 2017, Sánchez-Fernández et al., 2026, Peters and Skowron, 2020, Kalayci et al., 2025, Brill and Peters, 2023, Casey and Elkind, 2025]. These papers and others [Aziz et al., 2018a, Brill et al., 2024] have introduced a variety of voting rules to satisfy the JR axioms, including Thiele rules, the Method of Equal Shares (MES), and Phragmén, which we use in our work. See Lackner and Skowron [2023] for a survey of proportionality in multiwinner voting.

Following these lines of work, the justified representation axioms were generalized to the participatory budgeting setting in a series of papers. Aziz et al. [2018b] adapted JR and PJR to the PB setting in the case where a voter's utility for a project is equal to its cost. Peters et al. [2021] studied general additive utilities in PB: adapting EJR, introducing FJR, and proving that MES satisfies EJR for approval utilities. Subsequently, Los et al. [2022] extended the definition of PJR to additive utilities and proved that Phragmén satisfies PJR. Lastly, Brill et al. [2023] studied a restricted family of utility functions, showing positive and negative results for when the JR axioms can be satisfied for all of these utility functions simultaneously. For surveys on participatory budgeting, see Rey et al. [2025b] and Aziz and Shah [2020].

Participatory budgeting with constraints Several papers have studied participatory budgeting with constraints similar to ours. Jain et al. [2021] consider PB with project groups, where projects are partitioned into groups and each group has its own budget limit, and study the computational complexity of maximizing welfare. Hershkowitz et al. [2021] study the problem of maximizing social welfare in participatory budgeting under the constraint that each district in the city is entitled to at least some portion of the budget. Chen et al. [2022] study PB with donations, where voters can donate their own money to fund projects, and various axioms related to these donations. In their model they also allow project types and lower/upper quotas on the types. Rey et al. [2025a] propose a general framework for PB with constraints, and embed the problem into judgment aggregation. They focus on satisfying a set of monotonicity axioms. Goyal et al. [2023] also study a model of PB with project types, but focus on strategic and welfare considerations. Fairstein et al. [2021] study a model of PB with substitute projects, where voters can express which projects are substitutes. They also give an adaptation of MES, and show it satisfies an up-to-one relaxation of an adapted version of EJR. However, in their model each project has a single type, whereas our framework allows projects to belong to multiple types. Hence, both their model and axioms do not capture our setting with type constraints.

Proportionality under feasibility constraints Most closely related to our work are a series of papers studying proportionality axioms in participatory budgeting under general feasibility constraints [Mavrov et al., 2023, Masařík et al., 2024, Skowron, 2026]. In these papers, the PB instance comes with a set of feasible sets $\mathcal{F}$, which denote which outcomes are allowable. This general approach captures our problem, if we take $\mathcal{F}$ to be the set of outcomes that respect all type budgets. However, their model is also more general, allowing for more complicated feasibility constraints, such as lower quotas requiring that we have a certain number of projects from a specific type. Mavrov et al. [2023] introduces the restrained-EJR axiom as a generalization of EJR to this setting. Masařík et al. [2024] then introduced an alternative definition of EJR for this setting, proving that it implies restrained-EJR. Skowron [2026] then generalized this setting to one with additive utilities and generalized the latter definition of EJR. Although these works use the idea of candidate types as a motivating factor, we find that their axioms are too weak for our problem. See Example 1 and Example 2.

Separately, Motamed et al. [2022] study PB where projects have costs for multiple resources and each resource has a budget—a general model that can capture our problem. They introduce a proportionality axiom that uses a similar idea to ours, requiring voters to proportionally deserve enough budget in each resource. However, their axiom is considerably weaker than ours as it is only binding for groups of voters with identical preferences, whereas our axioms also guarantee representation to arbitrary voter groups that approve candidates in common.

Related Problems A special case of our model is that of dynamic multiwinner voting, where the goal is to update a previously selected or partially selected committee while limiting the changes made to the starting committee [Zech et al., 2024, Dong et al., 2026, Dong and Peters, 2025, Elkind et al., 2024, Cheng et al., 2020]. Another special case of our model is temporal voting, where there is a sequence of elections each electing one candidate and the aim is to choose a sequence of winners that satisfies JR-style axioms [Bulteau et al., 2021, Chandak et al., 2026, Elkind et al., 2025, Phillips et al., 2026, Teh, 2026]. The problem of maximizing voters' utilities in our model is closely related to multi-dimensional knapsack problems [Kellerer et al., 2004]. A number of papers have studied diversity and feasibility constraints in multiwinner voting with ordinal preferences [Bredereck et al., 2018, Celis et al., 2018]. Chingoma et al. [2024] study public decisions, a special case of PB with general feasibility constraints, introducing JR axioms that are tailored to their setting.

3 Preliminaries

In this section, we formally define the model, review existing proportionality axioms and their limitations in a simple special case of our setting, and then introduce our new proportionality axioms.

3.1 Participatory Budgeting with Type Constraints (τ-PB)

We consider approval-based participatory budgeting with type constraints. For each $d \in \mathbb{N}$, let $[d] = \{1, 2, \dots, d\}$. An *instance* is a tuple $\mathcal{I} = (N, C, \mathrm{cost}, (A_i)_{i \in N}, \mathcal{H}, (b_t)_{t \in [d]})$, where:

- $N = [n]$ and $C = \{c_1, \dots, c_m\}$ are the sets of n *voters* and m *candidates* (or *projects*);
- $\mathrm{cost} : C \to \mathbb{Q}_{>0}$ is a function that assigns to each candidate $c \in C$ its cost. For each $C' \subseteq C$, we write $\mathrm{cost}(C') = \sum_{c \in C'} \mathrm{cost}(c)$ for the total cost of C';
- for each voter $i \in N$, the set $A_i \subseteq C$ is the *approval set* of voter i;
- $\mathcal{H} = \{T_1, \dots, T_d\}$ is the *constraint structure*, where each $T_t \subseteq C$ is the set of candidates associated with type $t \in [d]$;
- for each $t \in [d]$, $b_t \in \mathbb{Q}_{>0}$ is the *type budget* for type t.

An *outcome* for $\mathcal{I}$ is a subset of candidates $W \subseteq C$. Given an outcome W and a type $t \in [d]$, let $W_t = W \cap T_t$ denote the set of selected candidates of type t. We say that W is *feasible* if $\mathrm{cost}(W_t) \leq b_t$ for every $t \in [d]$.

We assume that voters have approval utilities, i.e., each voter values an outcome according to the number of selected candidates that the voter approves: formally, the utility of a voter $i \in N$ from an outcome $W \subseteq C$ is $u_i(W) = |A_i \cap W|$. A *voting rule*, or simply a *rule*, is a function that maps each instance to a feasible outcome.

To illustrate the expressive power of our model, we show that it can capture three important settings.

Participatory budgeting and multiwinner voting. The classic model of approval-based participatory budgeting corresponds to $d = 1$, $T_1 = C$. Then an outcome $W \subseteq C$ is feasible if and only if $\mathrm{cost}(W) \leq b_1$. If furthermore every candidate has unit cost and $b_1 \in \mathbb{N}$, our model reduces to multiwinner voting.

Temporal voting. In temporal multiwinner voting with approval ballots, the election proceeds in ℓ rounds: in each round, the voters report approval preferences over the set of candidates P (the preferences may vary across rounds) and a single winner is selected. The aim is to choose a sequence of winners that satisfies JR-style axioms [Bulteau et al., 2021, Chandak et al., 2026, Elkind et al., 2025, Phillips et al., 2026, Teh, 2026]. Our model can represent this setting as follows. For each round $t \in [\ell]$ and each candidate $p \in P$, we create a unit-cost candidate $c_{p,t} \in C$, and for each round $t \in [\ell]$ let $T_t = \{c_{p,t} : p \in P\}$ and $b_t = 1$. Then a feasible outcome of size exactly ℓ selects exactly one candidate from each round.

Committee Renewal/Reconfiguration. In committee renewal, the goal is to update a previously selected committee so as to reflect the voters' evolving preferences while limiting the number of replaced candidates [Cheng et al., 2020, Bredereck et al., 2022, Elkind et al., 2024, Zech et al., 2024, Dong and Peters, 2025, Dong et al., 2026]. In this case, we are given an initial committee $W_0 \subseteq C$ of size k. We are then allowed to replace up to b candidates from W_0 for some $b \leq k$. To capture this setting within our model, we split the candidates into two types, by setting $T_1 = C, T_2 = C \setminus W_0$, and $b_1 = k, b_2 = b$. Then an outcome $W \subseteq C$ is feasible if and only if $|W| \leq k$ and $|W \setminus W_0| \leq b$. Since this setting uses only one type constraint in addition to the global budget constraint, we henceforth refer to it as *multiwinner voting with a single type constraint (*MW-SINGLE*).* Note that when $b = k$, MW-SINGLE reduces to multiwinner voting.

In what follows, we discuss existing axioms (and their natural variants) for PB, and show that negative results already arise in the special case of MW-SINGLE. For simplicity, we represent an instance of MW-SINGLE by the tuple $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$.

3.2 Standard Proportionality Axioms and Limitations

Most prior work on proportional representation in multiwinner voting and PB with approval ballots is based on the hierarchy of *justified representation* axioms [Aziz et al., 2017, Sánchez-Fernández et al., 2026, Peters et al., 2021].

Definition 1 (JR and EJR)**.** Given an instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ of MW-SINGLE and an $\ell \in [k]$, we say that a group of voters $S \subseteq N$ is *ℓ-cohesive* if $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\bigcap_{i \in S} A_i\right| \geq \ell$. A feasible committee W satisfies *ℓ-justified representation* (ℓ-JR) if, for every ℓ-cohesive group of voters $S \subseteq N$, there exists a voter $i \in S$ with $u_i(W) \geq \ell$. A feasible committee satisfies *justified representation* (JR) if it satisfies 1-JR; it satisfies *extended justified representation* (EJR) if it satisfies ℓ-JR for all $\ell \in [k]$. A rule $\mathcal{R}$ *satisfies* JR/EJR if it returns a feasible outcome satisfying JR/EJR on all instances; $\mathcal{R}$ *conditionally satisfies* JR/EJR if it returns a feasible outcome satisfying JR/EJR whenever one exists.

In the 'vanilla' multiwinner setting (i.e., $b = k$), committees satisfying JR or EJR always exist and can be computed in polynomial time [Aziz et al., 2018a, Peters and Skowron, 2020]. However, there are instances with $b < k$ where no feasible committee satisfies JR. Indeed, let $n = k = 2$, $A_1 = \{x\}$, $A_2 = \{y\}$, and set $C = \{u, w, x, y\}$, $W_0 = \{u, w\}$, $b = 1$. Then the only size-2 subset of C that satisfies JR is $\{x, y\}$, but it is not feasible since $b = 1$.

Moreover, deciding if there are any JR committees is computationally hard.

Theorem 1. *Deciding whether an instance of* MW-SINGLE *admits a feasible committee that satisfies JR is NP-complete, and $W[2]$-hard when parameterized by b.*

Proof. We reduce from SET COVER. Let $(U, \mathcal{S}, b)$ be an instance of SET COVER, where $U = \{u_1, \ldots, u_p\}$ is the universe, $\mathcal{S} = \{S_1, \ldots, S_q\}$ is a family of subsets of U, and b is an integer.

Given such an instance, we construct an election instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ as follows. For each element $u_i \in U$, create a voter $i \in N$. Thus, $N = [n] = [p]$. For each set $S_j \in \mathcal{S}$, create a candidate c_j. In addition, create k dummy candidates $d_1, \ldots, d_k$, where we set $k = n$. Hence, $C = \{c_1, \ldots, c_q\} \cup \{d_1, \ldots, d_k\}$. For each voter $i \in N$, define $A_i = \{c_j : u_i \in S_j\}$, that is, voter i approves exactly those candidates corresponding to sets that contain u_i. Finally, let the initial committee be $W_0 = \{d_1, \ldots, d_k\}$.

We now prove correctness. Since $\frac{n}{k} = 1$, every voter forms a group that deserves one candidate, provided that the voter approves at least one candidate. In our construction, no voter approves any dummy candidate. Therefore, a feasible committee W satisfies JR if and only if, for every voter $i \in N$, the committee W contains at least one candidate c_j such that $u_i \in S_j$.

Because W_0 consists entirely of dummy candidates, every non-dummy candidate included in W must replace one candidate from W_0. Since W is feasible, we have $|W \setminus W_0| \leq b$. It follows that W satisfies JR if and only if the non-dummy candidates selected by W correspond to a set cover of U of size at most b.

Therefore, there exists a feasible committee satisfying JR if and only if the given SET COVER instance admits a set cover of size at most b. The construction is polynomial. Since SET COVER is NP-hard and $W[2]$-hard when parameterized by b, the same holds for our problem. It is also known that JR can be verified in polynomial time [Aziz et al., 2017]. Hence, the problem is NP-complete. $\square$

Furthermore, even if W_0 already satisfies JR, it is hard to decide whether there exists a feasible committee that satisfies EJR.

Proposition 1. *Given an instance of* MW-SINGLE*, deciding whether a feasible EJR committee exists is NP-hard and is $W[2]$-hard when parameterized by b, even if the initial committee satisfies JR.*

Proof. We reduce from SET COVER. Let $(U, \mathcal{S}, b)$ be an instance of SET COVER, where $U = \{u_1, \ldots, u_p\}$ is the universe, $\mathcal{S} = \{S_1, \ldots, S_q\}$ is a family of subsets of U, and b is an integer.

Given such an instance, we construct an election instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ as follows.

For each element $u_i \in U$, create a block $B_i = \{v_i^1, \ldots, v_i^{q+1}\}$ of $q + 1$ voters. Thus, $n = p(q + 1)$. For each set $S_j \in \mathcal{S}$, create a candidate c_j. For each element u_i, create q private candidates $a_i^1, \ldots, a_i^q$. Finally, create p dummy candidates $d_1, \ldots, d_p$. Let $C = \{c_1, \ldots, c_q\} \cup \{a_i^r : i \in [p], r \in [q]\} \cup \{d_1, \ldots, d_p\}$ and set the committee size to $k = p(q + 1)$. Hence $\frac{n}{k} = 1$. For each voter $v_i^r \in B_i$, define $A_{v_i^r} = \{a_i^1, \ldots, a_i^q\} \cup \{c_j : u_i \in S_j\}$. Let the initial committee be $W_0 = \{a_i^r : i \in [p], r \in [q]\} \cup \{d_1, \ldots, d_p\}$. Then $|W_0| = pq + p = k$. Moreover, every voter approves m private candidates. Hence W_0 satisfies JR.

We show that the SET COVER instance has a cover of size at most b if and only if there exists a feasible committee satisfying EJR.

First, suppose that $\mathcal{S}' = \{S_j : j \in J\}$ is a set cover with $|J| \leq b$. Construct W by selecting all the private candidates, all candidates c_j with $j \in J$, and $p - |J|$ arbitrary dummy candidates. Then $|W| = k$ and $|W \setminus W_0| = |J| \leq b$, so W is feasible.

We claim that W satisfies EJR. Every voter gets utility at least q from its private candidates. Therefore, any group that deserves at most q candidates is already satisfied. It remains to consider groups that may deserve at least $q + 1$ candidates. Since $\frac{n}{k} = 1$, such a group must contain at least $q + 1$ voters. If the group contains voters from two different blocks, then its common approved candidates are only candidates c_j with $j \in J$, and there are at most q of them. Hence, it cannot deserve $q + 1$ candidates. Thus, the only possible group deserving $q + 1$ candidates must be some block B_i. Since $\mathcal{S}'$ covers U, for each $i \in [p]$ there exists $j \in J$ with $u_i \in S_j$. Hence, every voter in B_i approves all q private candidates of block B_i and also the selected candidate c_j. Thus every voter in B_i has utility at least $q + 1$, so EJR is satisfied.

Conversely, suppose that there exists a feasible committee W satisfying EJR. Fix an element u_i and consider the block B_i. The group B_i has size $q + 1$, and since $\frac{n}{k} = 1$, it can deserve $q + 1$ candidates. Indeed, because u_i belongs to some set S_j, the set $\{a_i^1, \ldots, a_i^q, c_j\}$ is commonly approved by all voters in B_i and has size $q + 1$. By EJR, some voter in B_i has utility at least $q + 1$ under W. This voter approves only the q private candidates of block i and the candidates c_j with $u_i \in S_j$. Since there are only q private candidates, W must contain at least one candidate c_j with $u_i \in S_j$. This holds for every element u_i, so these non-private candidates selected by W form a set cover of U. Finally, all non-private candidates lie outside W_0. Since W is feasible, $|W \setminus W_0| \leq b$, and therefore at most b non-private candidates are selected. Hence, the selected non-private candidates define a set cover of size at most b.

Thus, the constructed instance admits a feasible EJR committee if and only if the original SET COVER instance has a cover of size at most b. The reduction is polynomial, so the problem is NP-hard and $W[2]$-hard when parameterized by b, even when the initial committee satisfies JR. ☐

Theorem 1 implies that no polynomial-time computable voting rule conditionally satisfies JR (unless P=NP), since one can verify in polynomial time if a given committee satisfies JR [Aziz et al., 2017]. Moreover, we will now argue that no Thiele rule conditionally satisfies JR; this result is surprising, because this family of rules contains the Proportional Approval Voting (PAV) rule, which is NP-hard to compute and satisfies EJR in the standard setting [Aziz et al., 2017].

Definition 2 (Thiele rules)**.** A *weight function* is a mapping $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(1) = 1$ and $f(s) \geq f(s + 1)$ for every $s \in \mathbb{N}$. Given a feasible committee W and a weight function f, we define the f*-Thiele score* of W as $\mathrm{score}_f(W) = \sum_{i \in N} \sum_{s=1}^{|A_i \cap W|} f(s)$. The f-Thiele rule outputs feasible committees that maximize the f-Thiele score.

Proposition 2. *There exists an instance of* MW-SINGLE *that admits a JR committee for which no Thiele rule returns a committee satisfying JR.*

Proof. Consider the instance $\mathcal{I} = (N, C, (A_i)_{i \in N}, W_0, k, b)$ defined as follows. Let $C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, set $k = 4$, let $W_0 = \{c_1, c_2, c_3, c_4\}$, and set the replacement budget to $b = 2$. We create $n = 24$ voters. One voter approves $\{c_1, c_5\}$, five voters approve $\{c_5\}$, seven voters approve $\{c_6\}$, seven voters approve $\{c_7\}$, two voters approve $\{c_3\}$, and two voters approve $\{c_4\}$. Fix any Thiele weight function f with $f(1) = 1$.

Consider the feasible committee $W^\star = \{c_3, c_4, c_6, c_7\}$. Since $W^\star \setminus W_0 = \{c_6, c_7\}$, this committee is feasible. Its Thiele score is $\mathrm{Score}_f(W^\star) = 18$.

We now argue that every feasible committee satisfying JR has score at most 17. Any such committee must contain c_6, since the seven voters approving only c_6 form a group that deserves one candidate. Similarly, it must contain c_7. It must also intersect $\{c_1, c_5\}$, since six voters commonly approve c_5 and therefore deserve one candidate.

Since $c_5 \notin W_0$ and $c_6, c_7 \notin W_0$, feasibility forbids selecting c_5 together with both c_6 and c_7. Hence, any feasible JR committee must contain c_1. With c_1, c_6, c_7 fixed, only one slot remains. This slot can represent at most two additional voters, by choosing either c_3 or c_4. Therefore, every feasible JR committee represents at most $1 + 7 + 7 + 2 = 17$ voters, and hence has Thiele score at most 17.

Thus, every Thiele rule strictly prefers $W^\star$ to any feasible committee satisfying JR. $\square$

3.3 Type-Aware Proportionality Axioms

We have seen that there are instances of MW-SINGLE that admit no JR committees. The key reason for this is that in the standard PB model, a group of voters may claim representation whenever it can identify a set of commonly approved candidates X that it can afford using its proportional share of the *overall* budget; crucially, X need not be affordable with respect to the group's share of type budgets. However, intuitively, in the presence of constraints, a group should be regarded as deserving representation only if it can propose a set of candidates that fits within the group's proportional share of *every* type budget. We will now formalize this idea of 'deservingness' and the resulting notions of justified representation.

Definition 3. Given an instance $\mathcal{I}$ of τ-PB and an $\ell \in \mathbb{N}$, we say that a group of voters $S \subseteq N$ is *ℓ-deserving* if there exists a set $X \subseteq \bigcap_{i \in S} A_i$ such that $|X| \geq \ell$ and, for every type $t \in [d]$, we have $\mathrm{cost}(X \cap T_t) \leq b_t \cdot \frac{|S|}{n}$.

Proposition 3. *In the special case of τ-PB that corresponds to multiwinner voting (resp., PB), a group of voters is ℓ-deserving if and only if it is an ℓ-cohesive group in multiwinner voting (resp. PB).*

Proof. As multiwinner voting is a special case of PB, and previous work has shown that the definitions of ℓ-cohesive in the two setting coincide, we only need to show the equivalence between τ-PB and standard PB [Peters et al., 2021].

In this special case, there is only one type $t^\star \in [d]$ and it has $T_{t^\star} = C$ and $b_{t^\star} = b$, where b is the total PB budget. Hence, an outcome $W \subseteq C$ is feasible if and only if $\mathrm{cost}(W) \leq b$, which is exactly the feasibility condition allowed under standard PB.

Now fix a group $S \subseteq N$ and an integer $\ell \geq 1$. We show that S is ℓ-deserving in τ-PB if and only if it is ℓ-cohesive in standard PB.

If S is ℓ-deserving in τ-PB, then there exists a set $X \subseteq \bigcap_{i \in S} A_i$ such that $|X| \geq \ell$ and $\mathrm{cost}(X \cap T_{t^\star}) \leq b_{t^\star} \cdot \frac{|S|}{n}$. Using the specific structure of the instance, this gives $|S| \geq \mathrm{cost}(X) \cdot \frac{n}{b}$, and so S is ℓ-cohesive in standard PB.

Conversely, if S is ℓ-cohesive in standard PB, then there exists a set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq \ell$ and $|S| \geq \mathrm{cost}(X) \cdot \frac{n}{b}$. Thus, by the structure of our instance, we obtain $\mathrm{cost}(X \cap T_{t^\star}) \leq b_{t^\star} \cdot \frac{|S|}{n}$. Therefore, S is ℓ-deserving in τ-PB.

Thus, the two notions of deserving coincide, and the claim follows. $\square$

We can now use this notion of an ℓ-deserving group to generalize EJR to τ-PB. We note that the JR axiom, which we defined in the multiwinner setting, does not have a natural analog in the PB setting. We will also introduce an adaptation of the PJR axiom [Sánchez-Fernández et al., 2026, Los et al., 2022], which is slightly weaker than EJR (but stronger than JR in the multiwinner setting).

Definition 4 (Proportional/Extended Justified Representation with Type Constraints (τ-PJR/EJR)). Given a τ-PB instance $\mathcal{I}$, a feasible outcome W satisfies τ-PJR (resp., τ-EJR) if for every integer $\ell \geq 1$ and every ℓ-deserving group of voters $S \subseteq N$, it holds that $\left|W \cap \bigcup_{i\in S} A_i\right| \geq \ell$ (resp., there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$). A rule $\mathcal{R}$ satisfies τ-PJR (resp., τ-EJR) if all the outcomes returned by f satisfy τ-PJR (resp., τ-EJR).

We also recall the two extant EJR axioms by Mavrov et al. [2023] and Masařík et al. [2024], then state and analyze two examples, which illustrates that these axioms fail to adequately capture proportionality in PB with type constraints.

Restrained Extended Justified Representation (R-EJR). We refer to the Restrained Extended Justified Representation of in multiwinner elections with allocation constraints from Mavrov et al. [2023] as R-EJR. Let $\mathcal{P}$ be the family of feasible committees of size at most k. For $S \subseteq N$ and $T \subseteq C$, let $A_S(T) = (\bigcap_{i\in S} A_i) \cap T$. A committee $\widehat{W}$ is q-completable if there exists a set W'' with $|W''| \leq q$ such that $\widehat{W} \cup W'' \in \mathcal{P}$. A committee $W \in \mathcal{P}$ satisfies Restrained Extended Justified Representation (R-EJR) if there is no constraint-feasible blocking coalition $S \subseteq N$. Such a coalition has endowment $k_S = \lfloor \frac{|S|k}{n} \rfloor$ and satisfies that, for every k_S-completable set $\widehat{W} \subseteq W$ with $|\widehat{W}| \leq k - k_S$, there exists a set W' with $|W'| \leq k_S$ such that $\widehat{W} \cup W' \in \mathcal{P}$ and $|A_S(\widehat{W} \cup W')| \geq \max_{i\in S} u_i(W) + 1$.

Extended Justified Representation with Feasibility Constraints (EJR-FC). We refer to the feasibility constrained version of Extended Justified Representation from Masařík et al. [2024] as EJR-FC. Given an election $E = (C, N, \mathcal{P}, \mathcal{A})$ and an outcome $W \in \mathcal{P}$, a group $S \subseteq N$ deserves ℓ candidates in W if, for every set $T \subseteq W$, either there exists $X \subseteq \bigcap_{i\in S} A_i$ with $|X| \geq \ell$ such that $T \cup X \in \mathcal{P}$, or $\frac{|S|}{n} > \frac{\ell}{|T|+\ell}$. A feasible outcome W satisfies EJR-FC if every group S that deserves ℓ candidates in W contains a voter i with $u_i(W) \geq \ell$.

We will now give two examples and show that EJR-FC allows for the selection of outcomes which are undesirable with respect to proportional representation. Since EJR-FC strengthens R-EJR, it follows that these undesirable outcomes also satisfy R-EJR.

Example 1. *Consider the instance* $\mathcal{I} = (N, C, (A_i)_{i\in N}, W_0, k, b)$ *defined as follows. Let* $N = \{1, 2, 3, 4\}$ *and let* $C = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\}$*. Set* $k = 4$*,* $b = 2$*, and* $W_0 = \{a_1, b_1, c_1, c_2\}$*. The approval sets are* $A_1 = A_2 = \{a_1, a_2\}$ *and* $A_3 = A_4 = \{b_1, b_2\}$*. The candidates* c_1, c_2, c_3, c_4 *are approved by no voter.*

Consider the two feasible committees $W = \{a_1, b_1, c_3, c_4\}$ and $W^\star = \{a_1, a_2, b_1, b_2\}$.

We first observe that W violates standard EJR. For $S = \{1, 2\}$, we have $|S| = 2 \cdot \frac{n}{k}$ and $\bigcap_{i\in S} A_i = \{a_1, a_2\}$. Thus S is 2-cohesive, but both voters in S obtain utility only 1 from W. The same argument applies to $\{3, 4\}$. In contrast, $W^\star$ satisfies standard EJR, since voters $1, 2$ obtain a_1, a_2, and voters $3, 4$ obtain b_1, b_2.

We next show that W satisfies EJR-FC. The only groups with nonempty common approval are subsets of $\{1, 2\}$ and subsets of $\{3, 4\}$. By symmetry, it suffices to consider groups contained in $\{1, 2\}$. For $S = \{1, 2\}$, the only possible improvement is to obtain two commonly approved candidates. For EJR-FC, take $T = \{c_3, c_4\}$. Since c_3 and c_4 already exhaust the replacement budget, no feasible completion can contain a_2, and hence no completion can give S both candidates in $A_1 \cap A_2 = \{a_1, a_2\}$. Hence, there does not exist any feasible $T \cup X$ with $X \subseteq A_1 \cap A_2$ and $|X| \geq 2$. Moreover, $\frac{|S|}{n} = \frac{1}{2}$ and $\frac{2}{|T|+2} = \frac{1}{2}$, so the strict inequality in the definition also fails. Hence S does not deserve 2 candidates in W, and is thus satisfied with respect to EJR-FC.

For a singleton group $S = \{1\}$ or $S = \{2\}$, take $T = \{b_1, c_3, c_4\}$. Since c_3 and c_4 already exhaust the replacement budget, no feasible completion can contain both a_1 and a_2. Furthermore, the required inequality would be $\frac{1}{4} > \frac{2}{3+2}$, which is false. Thus no singleton group blocks under EJR-FC either, and it follows that W satisfies EJR-FC. Hence, the EJR axioms by Mavrov et al. [2023] and Masařík et al. [2024] are satisfied by both of the committees W and $W^\star$ in Example 1

Lastly, note that W uses the type budget to buy dummy candidates approved by no voter, despite the existence of candidates approved by underrepresented voters, which $W^\star$ selects in contrast. Accordingly, $W^\star$ satisfies EJR and our axiom, τ-EJR, while W does not.

Example 2. *Consider an instance of multiwinner voting with type constraints in which there are two disjoint types* $T_1 = \{c_1, c_2, c_3\}$ *and* $T_2 = \{c_4, c_5, c_6, c_7\}$ *with type budgets* $b_1 = 2$ *and* $b_2 = 4$. *Suppose half of the voters,* $S \subseteq N$, *approve of candidates* $\{c_1, c_4, c_5\}$.

Since the types are disjoint and thus their constraints do not interact, we can think of this case as running two elections in parallel. Intuitively, since S constitutes half of the voters and every voter in S has identical preferences, they should be entitled to decide on half of the candidates from the first type, and half of the candidates from the second type. So S should be entitled to a representation of three candidates.

Now, consider the committee $W = \{c_2, c_3, c_4, c_5, c_6, c_7\}$. This committee contains only two candidates that represent the group S. However, this committee satisfies EJR-FC, and thus R-EJR.In contrast, our definitions require that S be given three representatives on this example.

We now describe in detail why W satisfies EJR-FC, and thus R-EJR. Since the group S already approves two representatives in W, it suffices to ensure that S is not entitled to three representatives. In order for S to deserve three representatives, it must hold that for every $T \subseteq W$ there exists an $X \subseteq \bigcap_{i \in S} A_i$ with $|X| \geq 3$ such that either $T \cup X$ is feasible or $|S| > \frac{3n}{3+|T|}$. Since the intersection of approvals of S has size three, the only possible such X is $\{c_1, c_4, c_5\}$. Now, consider $T = \{c_2, c_3\}$. We note that $T \cup X$ is not feasible as this would exceed the type-1 budget of 2. And the second requirement says that we need $|S| > \frac{3n}{3+2} = \frac{3n}{5}$. Since $|S| = \frac{n}{2}$, it does not meet this requirement. Thus, S is not deserving of three or more candidates, and W satisfies EJR-FC.

Our next challenge is to identify voting rules that satisfy our axioms. A natural candidate is the PAV rule, which is known to satisfy the EJR axioms of Mavrov et al. [2023] and Masařík et al. [2024]. However, it turns out that τ-PJR (and hence τ-EJR) is not satisfied by any Thiele rule (including PAV), even in the MW-SINGLE setting.

Proposition 4. *For each weight function* f, *there exists an instance of* MW-SINGLE *on which the* f*-Thiele rule does not return a committee satisfying* τ*-PJR.*

Proof. Fix any weight function w. We distinguish two cases.

Case 1. $w(s) = 1$ for all $s \geq 1$.

Then the rule is Approval Voting. We give a counterexample.

Let $k = b = 6$. Let $W_0 = \{a_1, a_2, z_1, z_2, z_3, z_4\}$. There are 24 voters. A group S of 8 voters approves exactly $\{a_1, a_2, a_3, a_4\}$. The remaining 16 voters approve exactly $\{c_1, \ldots, c_6\}$.

Now $X = \{a_1, a_2\} \subseteq \cap_{i \in S} A_i$ satisfies $|X| = 2$ and $|X \setminus W_0| = 0 \leq b \cdot \frac{|S|}{n}$. Hence, S is 2-deserving.

However, each c_j is approved by 16 voters, while a_1 and a_2 are approved by only 8 voters. Therefore, the Thiele winner is $\{c_1, \ldots, c_6\}$. Thus $|W \cap (\cup_{i \in S} A_i)| = 0 < 2$, so τ-PJR is violated.

Case 2. There exists some $\ell \geq 2$ such that $w(\ell) < 1$. Let ℓ be the smallest such index.

Choose an integer $b \geq 2$ and positive integers s, t such that $w(\ell) < \frac{t}{s} < \frac{b-1}{b}$.

Let $n = s + bt$ and $k = \ell b > \ell \cdot (1 + \frac{bt}{s}) = \frac{\ell n}{s}$. Then $s > \ell \cdot \frac{n}{k}$.

The initial committee W_0 consists of $\ell - 1$ candidates $d_1, \ldots, d_{\ell-1}$ and $k - \ell + 1$ dummy candidates approved by no one. Outside W_0, there are $b + 1$ candidates $x, y_1, \ldots, y_b$.

The voters are partitioned into one group S of size s and b groups $T_1, \ldots, T_b$, each of size t. Every voter in S approves exactly $\{d_1, \ldots, d_{\ell-1}, x\}$. Every voter in T_j approves exactly $\{y_j\}$.

We first show that S is ℓ-deserving. Indeed, $|S| = s > \ell \cdot \frac{n}{k}$. Also, $X = \{d_1, \ldots, d_{\ell-1}, x\} \subseteq \cap_{i \in S} A_i$ has size ℓ and satisfies $|X \setminus W_0| = 1$. Since $\frac{t}{s} < \frac{b-1}{b}$, we have $bt < (b-1)s$, so $n = s + bt < bs$. Hence, $1 < b \cdot \frac{s}{n} = b \cdot \frac{|S|}{n}$. Therefore, S is ℓ-deserving.

Now let W be a winning feasible committee. Since the dummy candidates in W_0 are approved by no one, every winner must contain $d_1, \ldots, d_{\ell-1}$, because replacing any dummy candidate by a missing d_h preserves feasibility and strictly increases the score.

Algorithm 1 Method of Equal Shares for Type Constraints (τ-MES)

```
$W \leftarrow \emptyset$
for all $i \in N$ and all $t \in [d]$: $p_i^{(t)} \leftarrow \frac{b_t}{n}$
**while** true **do**
    **for** each $c \in C \setminus W$ and each $t \in [d]$ **do**
        Define the type price as follows, if the minimum exists
        $\eta_t(c) = \begin{cases} 0, & c \notin T_t, \\ \min\Big\{\eta \geq 0 : \sum_{i \in N : c \in A_i} \min\{\eta, p_i^{(t)}\} \geq \operatorname{cost}(c)\Big\}, & c \in T_t, \end{cases}$
    **end for**
    Call $c$ affordable if $\eta_t(c)$ exists for every $t \in [d]$
    If no affordable candidate exists, **break**
    Pick an affordable candidate $c^\star \in \arg\min_{c \in C \setminus W} \max_{t \in [d]} \eta_t(c)$, breaking ties arbitrarily
    **for** each $t \in [d]$ with $c^\star \in T_t$ **do**
        Update $p_i^{(t)} \leftarrow p_i^{(t)} - \min\{\eta_t(c^\star), p_i^{(t)}\}$ for all $i \in N$ s.t. $c^\star \in A_i$
    **end for**
    $W \leftarrow W \cup \{c^\star\}$
**end while**
**return** $W$
```

Suppose for contradiction that $x \in W$. Since at most b candidates outside W_0 can be chosen and there are $b+1$ such candidates, there exists some j such that $y_j \notin W$. Define $W' = (W \setminus \{x\}) \cup \{y_j\}$. Then $W' \in \mathcal{F}$.

Because W contains $d_1, \ldots, d_{\ell-1}$, every voter in S has exactly ℓ approved candidates in W and exactly $\ell - 1$ approved candidates in W'. Thus removing x decreases the total score by exactly $s \cdot w(\ell)$. On the other hand, adding y_j increases the total score by exactly t. Since $\frac{t}{s} > w(\ell)$, $\operatorname{score}_w(W') > \operatorname{score}_w(W)$, contradicting the optimality of W.

Hence no winning committee contains x. Since the voters in S approve only $\{d_1, \ldots, d_{\ell-1}, x\}$, every winner W satisfies $|W \cap (\cup_{i \in S} A_i)| \leq \ell - 1$. But S is ℓ-deserving. So W violates τ-PJR.

In both cases, the Thiele rule fails τ-PJR. ☐

Indeed, one may wonder if τ-PJR/τ-EJR are simply too demanding. We will now argue that this is not the case, by putting forward polynomial-time computable voting rules that satisfy these axioms in a range of settings that extend MW-SINGLE.

4 Main Results

Despite the previous section's negative results in the very restricted special case of multiwinner voting with a single type constraint, in this section, we present a polynomial-time algorithm that guarantees τ-PJR/τ-EJR for two natural generalizations of that setting. We present our algorithm in Section 4.1, show that it satisfies τ-PJR in τ-PB when the constraint sets form a laminar set family (Section 4.2), and prove that it satisfies τ-EJR in multiwinner voting when the constraint sets are nested (Section 4.3). Each of these algorithmic results is shown to be tight. In Section 4.4, we show that a property even stronger than τ-EJR is guaranteed to exist in τ-PB with arbitrary constraint sets, but that it is hard to compute an outcome satisfying even τ-PJR in this most general setting.

4.1 The Method of Equal Shares with Type Constraints (τ-MES)

The Method of Equal Shares (MES) has recently emerged as one of the most widely studied rules in approval-based participatory budgeting and multiwinner voting [Peters and Skowron, 2020, Peters et al., 2021]. In this subsection, we introduce τ-MES, an extension of MES to the setting of τ-PB.

At a high level, τ-MES assigns each voter an equal share of every type budget. A candidate is affordable if and only if its approving voters can collectively cover its cost within every type to which it belongs. The rule then selects an affordable candidate for which the maximum payment required from its supporters for any type is minimized, subject to payments for a candidate being as equal as possible across its supporters.

Before proving the proportionality guarantees of τ-MES, we prove that it returns feasible outcome.

Lemma 1. *Given an instance of τ-PB, τ-MES always returns a feasible outcome.*

Proof. Consider an outcome W output by τ-MES on an instance $\mathcal{I}$ of τ-PB. To prove that the outcome is feasible it suffices to show that for every type $t \in [d]$, the budget constraint for t is met: $\text{cost}(W_t) \leq b_t$. When τ-MES added a candidate to W, it split the cost of the candidate across its approvers subject to their remaining budget limitations. Thus, we know that for any type t, $\text{cost}(W_t)$ is bounded by the combined type t budgets of all voters. Therefore, $\text{cost}(W_t) \leq n \cdot \frac{b_t}{n} = b_t$. $\square$

4.2 Laminar Types: τ-MES Satisfies τ-PJR

In this subsection, we show that τ-MES guarantees τ-PJR under laminar constraint structures. We say that the constraint structure $\mathcal{H} = \{T_1, \ldots, T_d\}$ is *laminar* if, for every $t, t' \in [d]$, either $T_t \subseteq T_{t'}$, or $T_{t'} \subseteq T_t$, or $T_t \cap T_{t'} = \emptyset$. This naturally captures any type structure which can be represented hierarchically, for example, spending limits both for neighborhoods and for candidate categories and sub-categories within those neighborhoods. Laminar candidate groupings have been previously studied in the context of PB [Goyal et al., 2023] and multiwinner voting [Bredereck et al., 2018]. We now present our most general polynomial-time algorithmic result.

Theorem 2. *In τ-PB with laminar type constraints, τ-MES satisfies τ-PJR.*

Proof. Fix $\ell \geq 1$ and an ℓ-deserving group $S \subseteq N$. By definition, there exists a set $X \subseteq \bigcap_{i \in S} A_i$ such that $|X| \geq \ell$ and $\text{cost}(X \cap T_t) \leq b_t \cdot \frac{|S|}{n}$ for every $t \in [d]$. By discarding extra candidates, we may assume that $|X| = \ell$. For simplicity, write $X_t = X \cap T_t$.

Suppose, towards a contradiction, that $\left|W \cap \bigcup_{i \in S} A_i\right| < \ell$. Let $y_1, \ldots, y_r$ be the candidates in $W \cap \bigcup_{i \in S} A_i$, listed in the order in which the algorithm selects them. Then $r < \ell$.

For each $j \in \{0, \ldots, r\}$, each $i \in S$, and each $t \in [d]$, let $q_{i,j}^{(t)}$ be voter i's remaining type-t balance immediately after y_j is selected, with the convention $q_{i,0}^{(t)} = \frac{b_t}{n}$. Since $y_1, \ldots, y_r$ are exactly the selected candidates approved by at least one voter in S, any selected candidate after y_{j-1} and before y_j is approved by no voter in S, and hence does not change these balances.

We define the residual witness sets $R_r \subseteq \cdots \subseteq R_1 \subseteq R_0 \subseteq X$ recursively. Intuitively, for each $j \in \{0, \ldots, r\}$, the set R_j consists of the candidates in the original witness set X that have not yet been matched to the selected candidates $y_1, \ldots, y_j$. We start with $R_0 = X$. Now fix $j \in \{1, \ldots, r\}$, and define R_j from R_{j-1} as follows.

- If $y_j \in R_{j-1}$, set $x_j = y_j$ and define $R_j = R_{j-1} \setminus \{x_j\}$.
- If $y_j \notin R_{j-1}$, define $\Lambda_j = \{\, t : y_j \in T_t \text{ and } R_{j-1} \cap T_t \neq \emptyset \,\}$. Thus, Λ_j is the set of types that contain y_j and intersect the residual set R_{j-1}.
  - If $\Lambda_j = \emptyset$, set $R_j = R_{j-1}$.
  - If $\Lambda_j \neq \emptyset$, then every set T_t with $t \in \Lambda_j$ contains y_j. Since $\mathcal{H}$ is laminar, these sets form a chain under inclusion. Let $u_j \in \Lambda_j$ be such that $T_{u_j} \subseteq T_t$ for every $t \in \Lambda_j$. Choose any $x_j \in R_{j-1} \cap T_{u_j}$, and define $R_j = R_{j-1} \setminus \{x_j\}$.

For each $j \in \{0, \ldots, r\}$ and each type $t \in [d]$, write $(R_j)_t = R_j \cap T_t$.

Claim 1. *For every $j \in \{0, \ldots, r\}$, the following properties hold:*

(i) $R_j \cap \{y_1, \ldots, y_j\} = \emptyset$

(ii) $|R_j| \geq \ell - j$

(iii) for every $i \in S$ *and every* $t \in [d]$, $q_{i,j}^{(t)} \geq \frac{\mathrm{cost}((R_j)_t)}{|S|}$

Proof of Claim. We proceed by induction on j. For $j = 0$, (i) and (ii) are immediate from $R_0 = X$ and $|X| = \ell$. Also, since $\mathrm{cost}(X_t) \leq b_t \cdot \frac{|S|}{n}$, we have $q_{i,0}^{(t)} = \frac{b_t}{n} \geq \frac{\mathrm{cost}(X_t)}{|S|} = \frac{\mathrm{cost}((R_0)_t)}{|S|}$, so (iii) holds.

Now assume the claim holds for $j - 1$. Properties (i) and (ii) follow directly from the construction of R_j. At step j we remove at most one candidate, and any removed candidate lies in R_{j-1}, hence is not among $y_1, \ldots, y_{j-1}$ by (i) for $j - 1$. It remains to prove property (iii). We proceed by considering the three cases in the definition of R_j. In each case we make use of the fact that no voter in S contributes to buying any candidate between y_{j-1} and y_j so we only have to reason about the change in remaining budgets following the purchase of y_j.

Case 1: $y_j \in R_{j-1}$**.**

In this case, by definition, $x_j = y_j$ and $R_j = R_{j-1} \setminus \{y_j\}$. Since $y_j \in R_{j-1} \subseteq X \subseteq \bigcap_{i \in S} A_i$, every voter in S approves y_j.

Fix a type $t \in [d]$. If $y_j \notin T_t$, then $\eta_t(y_j) = 0$ by definition. Thus, type-t balances are not changed when passing from R_{j-1} to R_j. Therefore $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$ and $(R_j)_t = (R_{j-1})_t$ and property (iii) follows by the induction hypothesis.

If $y_j \in T_t$, then $y_j \in (R_{j-1})_t$, and so property (iii) for $j - 1$ gives $q_{i,j-1}^{(t)} \geq \frac{\mathrm{cost}((R_{j-1})_t)}{|S|} \geq \frac{\mathrm{cost}(y_j)}{|S|}$ for all $i \in S$. Thus, since the voters in S all approve y_j, they could pay for y_j alone by each paying at most $\frac{\mathrm{cost}(y_j)}{|S|}$ in type t. Therefore $\eta_t(y_j) \leq \frac{\mathrm{cost}(y_j)}{|S|}$ for every $t \in [d]$. Hence, we have $q_{i,j}^{(t)} \geq q_{i,j-1}^{(t)} - \eta_t(y_j) \geq \frac{\mathrm{cost}((R_{j-1})_t)}{|S|} - \frac{\mathrm{cost}(x_j)}{|S|} = \frac{\mathrm{cost}((R_j)_t)}{|S|}$, where the second inequality uses property (iii) for $j - 1$, and the equality uses $x_j \in T_t$ together with $R_j = R_{j-1} \setminus \{y_j\}$. Thus, property (iii) holds.

Case 2: $y_j \notin R_{j-1}$ **and** $\Lambda_j = \emptyset$**.**

In this case, by definition, $R_j = R_{j-1}$. Fix $t \in [d]$. If $y_j \notin T_t$, then type t is not charged at step j, so $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$. Since also $R_j = R_{j-1}$, property (iii) follows immediately from the induction hypothesis.

If $y_j \in T_t$, then the definition of Λ_j and the assumption $\Lambda_j = \emptyset$ imply that $R_{j-1} \cap T_t = \emptyset$. Since $R_j = R_{j-1}$, we also have $(R_j)_t = \emptyset$. Therefore $q_{i,j}^{(t)} \geq 0 = \frac{\mathrm{cost}((R_j)_t)}{|S|}$ and property (iii) holds.

Case 3: $y_j \notin R_{j-1}$ **and** $\Lambda_j \neq \emptyset$**.**

In this case, by definition, we choose $u_j \in \Lambda_j$ such that $T_{u_j} \subseteq T_t$ for every $t \in \Lambda_j$, then choose $x_j \in R_{j-1} \cap T_{u_j}$, and finally set $R_j = R_{j-1} \setminus \{x_j\}$. Since $x_j \in R_{j-1} \subseteq X$ and $X \subseteq \bigcap_{i \in S} A_i$, every voter in S approves x_j.

Consider any type $u \in [d]$. If $x_j \notin T_u$, then $\eta_u(x_j) = 0$ by definition. If $x_j \in T_u$, then $x_j \in (R_{j-1})_u$, so property (iii) for $j - 1$ gives $q_{i,j-1}^{(u)} \geq \frac{\mathrm{cost}((R_{j-1})_u)}{|S|} \geq \frac{\mathrm{cost}(x_j)}{|S|}$ for all $i \in S$. Since every voter in S approves x_j, the voters in S can pay for x_j alone by each paying at most $\frac{\mathrm{cost}(x_j)}{|S|}$ in type u.

Since this holds for any type, we have $\eta_u(x_j) \leq \frac{\mathrm{cost}(x_j)}{|S|}$ for every $u \in [d]$. Since $x_j \in R_{j-1}$ while $y_j \notin R_{j-1}$, we know $x_j \neq y_j$. Hence, by property (i) for $j - 1$, x_j was still unselected when y_j was chosen. Since x_j was affordable and unselected at that moment, and the algorithm chooses y_j so as to minimize the price among all affordable unselected candidates, we conclude that $\max_{u \in [d]} \eta_u(y_j) \leq \max_{u \in [d]} \eta_u(x_j) \leq \frac{\mathrm{cost}(x_j)}{|S|}$.

Now fix $t \in [d]$. If $x_j \in T_t$, then using the above inequality we have $q_{i,j}^{(t)} \geq q_{i,j-1}^{(t)} - \eta_t(y_j) \geq \frac{\mathrm{cost}((R_{j-1})_t)}{|S|} - \frac{\mathrm{cost}(x_j)}{|S|} = \frac{\mathrm{cost}((R_j)_t)}{|S|}$.

Assume next that $x_j \notin T_t$. If $y_j \notin T_t$, then type t is not charged at step j, and since also $x_j \notin T_t$, removing x_j does not change the type-t part of the residual set. Thus $q_{i,j}^{(t)} = q_{i,j-1}^{(t)}$ and $(R_j)_t = (R_{j-1})_t$, so property (iii) follows immediately from the induction hypothesis.

Finally, suppose that $y_j \in T_t$. We claim that $R_{j-1} \cap T_t = \emptyset$. Indeed, if $R_{j-1} \cap T_t \neq \emptyset$, then by the definition of Λ_j we would have $t \in \Lambda_j$, because $y_j \in T_t$ and $R_{j-1} \cap T_t \neq \emptyset$. Since u_j was chosen so that $T_{u_j} \subseteq T_{t'}$ for every $t' \in \Lambda_j$, it would follow in particular that $T_{u_j} \subseteq T_t$. But $x_j \in T_{u_j}$ by construction, so this would imply $x_j \in T_t$, contradicting the assumption of the present case. Hence $R_{j-1} \cap T_t = \emptyset$. Since also $x_j \notin T_t$, removing x_j does not change the type-t part of the residual set, and therefore $(R_j)_t = \emptyset$. Consequently, $q_{i,j}^{(t)} \geq 0 = \frac{\mathrm{cost}((R_j)_t)}{|S|}$. Thus property (iii) holds in Case 3 as well.

This completes the induction for property (iii), and thus the proof of the claim. □

By Claim 1 (ii), for $j = r$, we have $|R_r| \geq \ell - r > 0$. Choose any $x \in R_r$. Since $R_r \subseteq X \subseteq \bigcap_{i\in S} A_i$, every voter in S approves x. We claim that $x \notin W$. Indeed, if $x \in W$, then $x \in W \cap \bigcup_{i\in S} A_i$, so by the definition of $y_1, \ldots, y_r$ we would have $x \in \{y_1, \ldots, y_r\}$, contradicting Claim 1 (i) for $j = r$.

Let $\bar{p}_i^{(t)}$ denote voter i's final type-t balance when the algorithm terminates. Since no candidate selected after y_r is approved by any voter in S, the balances of voters in S never change after the selection of y_r, and thus $\bar{p}_i^{(t)} = q_{i,r}^{(t)}$ for every $i \in S$ and $t \in [d]$. If x belongs to no type, then $\eta_t(x) = 0$ for every $t \in [d]$, so x is affordable at termination, contradicting the stopping condition.

Thus fix any $t \in [d]$ such that $x \in T_t$. Because $x \in (R_r)_t$, Claim 1 (iii) for $j = r$ gives $q_{i,r}^{(t)} = \bar{p}_i^{(t)} \geq \frac{\mathrm{cost}((R_r)_t)}{|S|} \geq \frac{\mathrm{cost}(x)}{|S|}$ for all $i \in S$. Since every voter in S approves x, we have $\sum_{i\colon x\in A_i} \min\left\{\frac{\mathrm{cost}(x)}{|S|}, \bar{p}_i^{(t)}\right\} = \sum_{i\in S} \min\left\{\frac{\mathrm{cost}(x)}{|S|}, \bar{p}_i^{(t)}\right\} = \sum_{i\in S} \frac{\mathrm{cost}(x)}{|S|} = \mathrm{cost}(x)$.

Thus $\eta_t(x)$ exists. Since t was arbitrary among the types containing x, the same argument shows that $\eta_t(x)$ exists for every type t with $x \in T_t$. For types t with $x \notin T_t$, we have $\eta_t(x) = 0$ by definition. Hence x is affordable when the algorithm terminates, contradicting the stopping condition.

This contradiction shows that $\left|W \cap \bigcup_{i\in S} A_i\right| \geq \ell$, and thus W satisfies τ-PJR. □

Since MES satisfies EJR in the approval-based PB setting without additional constraints [Peters et al., 2021], one might hope that the guarantee of Theorem 2 could be strengthened to τ-EJR. We now show that such a guarantee is not possible for any polynomial-time rule under laminar type constraints, unless P = NP.

Theorem 3. *Unless P = NP, no polynomial-time rule satisfies τ-EJR in τ-PB with laminar type constraints, even in the special case of unit costs*[2] *and pairwise disjoint types.*

Proof. We reduce from the following PCP hardness result for k-uniform hypergraph vertex cover [Dinur et al., 2003].[3] There exist a fixed integer k and constants $0 < \alpha < \beta < 1$ with $\beta - \alpha > \frac{1}{2}$ such that, given an k-uniform hypergraph $G = (V, E)$ with $|V|$, it is NP-hard to distinguish between the following two cases.

- G has a vertex cover of size at most $\alpha \cdot |V|$.
- Every vertex cover of G has size at least $\beta \cdot |V|$.

Choose a rational number γ with $\frac{1}{2} < \gamma = \frac{s}{q} < \beta - \alpha$, where $s, q \in \mathbb{N}^+$. Given G, create one voter for each vertex, so $n = |V|$. For every hyperedge $e \in E$ and every vertex $v \in e$, create a candidate

---

[2] We note that we reduce to instances that are more general than multiwinner voting since we allow for non-integral type budgets. Thus, this theorem does not rule out the possibility of a polynomial-time rule that satisfies τ-EJR in multiwinner voting with laminar type constraints.

[3] This reduction construction was discovered with the assistance of ChatGPT. The authors independently verified the resulting proof and refined its presentation. The authors take full responsibility for the correctness of the proof.

$c_{e,v}$. Voter u approves $c_{e,v}$ if and only if $u \neq v$. Thus, each candidate is disapproved by exactly one voter.

For each hyperedge e, create a type set $T_e = \{c_{e,v} : v \in e\}$. These types are pairwise disjoint, hence laminar. All candidates have unit cost, and each type has budget $b_e = \frac{1}{\gamma}$. Since $\gamma > \frac{1}{2}$, we have $\frac{1}{\gamma} < 2$, so every feasible outcome selects at most one candidate from each type. Let $\lambda = |E|$. Hence every feasible outcome has size at most λ.

Now suppose a polynomial-time rule always outputs a feasible outcome satisfying τ-EJR. Given G, construct the instance above and run the rule to obtain W. Accept if and only if W selects exactly one candidate from every type set T_e and more than $(1-\beta)n$ voters approve all candidates in W. Below, we show that every τ-EJR outcome passes this test in the YES case, whereas no τ-EJR outcome passes it in the NO case. Thus, any polynomial-time rule that always returns a τ-EJR outcome would distinguish these two cases of the underlying PCP problem for k-uniform hypergraph vertex cover, which incurs a contradiction.

Suppose first that G has a vertex cover $C^\star \subseteq V$ with $|C^\star| \leq \alpha \cdot |V| = \alpha n$. For each edge $e \in E$, fix exactly one representative $v_e \in C^\star \cap e$. Define $X^\star = \{c_{e,v_e} : e \in E\}$. Thus, $X^\star$ contains exactly one candidate for each edge e. Let $S^\star = V \setminus C^\star$. Then $|S^\star| \geq (1-\alpha)n$, and every voter in $S^\star$ approves all candidates in $X^\star$. Without loss of generality, set $r = \gamma n$.[4] Let W be any feasible outcome satisfying τ-EJR, and define $H_W = \{u \in S^\star : |A_u \cap W| \geq \lambda\}$. If $|H_W| \leq |S^\star| - r$, then $S^\star \setminus H_W$ contains a subgroup R of size r. This group is λ-deserving, but no voter in it obtains λ approved selected candidates, contradicting τ-EJR. Hence, $|H_W| \geq |S^\star| - r + 1$. Since $|S^\star| \geq (1-\alpha)n$ and $r = \gamma n$, it follows that $|H_W| \geq (1-\alpha-\gamma)n + 1 > (1-\beta)n$. Since every feasible outcome has size at most λ, any voter with at least λ approved selected candidates must approve all candidates in W with $|W| = \lambda$. Therefore, W selects exactly one candidate from each type e. Define $C_W = \{v \in V : \text{there exists } e \in E \text{ with } c_{e,v} \in W\}$. Then C_W intersects every hyperedge, so it is a vertex cover. Moreover, a voter u approves all candidates in W if and only if $u \notin C_W$. Since more than $(1-\beta)n$ voters approve all candidates in W, we get $|C_W| < \beta n$.

Conversely, suppose G is a NO instance, so every vertex cover has size at least βn. For any feasible outcome W, if W selects exactly one candidate from each type, then the set $C_W = \{v \in V : \text{there exists } e \in E \text{ with } c_{e,v} \in W\}$ is a vertex cover and has size at least βn. Thus, at most $(1-\beta)n$ voters approve all candidates in W, which fails the second condition in the verification procedure. Otherwise, if W does not select exactly one candidate from each type, then it fails the first condition in the verification procedure. $\square$

Given the impossibility of an efficient algorithm for τ-EJR under laminar type constraints, a natural next question is whether τ-MES can guarantee τ-PJR beyond laminar type constraints. We answer this question negatively by showing that any non-laminar constraint structure can be embedded in an instance of τ-PB such that τ-MES violates τ-PJR on the resulting instance.

Proposition 5. *Let C be a candidate set, and let $\mathcal{H}$ be a non-laminar constraint structure on C. Then there exists an instance $\mathcal{I} = (N, C, \text{cost}, (A_i)_{i \in N}, \mathcal{H}, (b_t)_{t \in [d]})$ of τ-PB for which the outcome returned by τ-MES does not satisfy τ-PJR.*

Proof. Since $\mathcal{H}$ is not laminar, there exist two types $T_1, T_2 \in \mathcal{H}$ and candidates x, y, z such that $x \in T_1 \setminus T_2$, $y \in T_2 \setminus T_1$, $z \in T_1 \cap T_2$. All candidates in $C \setminus \{x, y, z\}$ will be dummy candidates. Set the candidate costs as $\text{cost}(x) = \text{cost}(y) = \frac{2}{5}$, $\text{cost}(z) = \frac{1}{2}$, and give every dummy candidate cost 1. Set the type budgets by $b_1 = b_2 = 1$, and, for every other type $r \in [d] \setminus \{1, 2\}$, set $b_r = 2$. Let $N = \{1, 2, 3, 4, 5\}$. Define the approval sets by $A_1 = A_2 = \{x, y, z\}$, $A_3 = A_4 = A_5 = \{z\}$.

We first show that the group $S = \{1, 2\}$ is 2-deserving. Let $X = \{x, y\}$. Then $X \subseteq A_1 \cap A_2$ and $|X| = 2$. For type 1, we have $\text{cost}(X \cap T_1) = \text{cost}(x) = \frac{2}{5} = b_1 \cdot \frac{|S|}{n}$. Similarly, for type 2, we have $\text{cost}(X \cap T_2) = \text{cost}(y) = \frac{2}{5} = b_2 \cdot \frac{|S|}{n}$. Finally, for any other type $r \in [d] \setminus \{1, 2\}$, we have $\text{cost}(X \cap T_r) \leq \text{cost}(x) + \text{cost}(y) = \frac{4}{5} = b_r \cdot \frac{|S|}{n}$. Thus, S is 2-deserving.

---

[4] Since $\gamma = \frac{s}{q}$ with $s, q \in \mathbb{N}^+$, we may assume that q divides n. Otherwise, before constructing the voting instance, replace G by the disjoint union of q copies of G.

We now analyse the execution of τ-MES. Initially, every voter has type-1 and type-2 balance $\frac{1}{5}$, and balance $\frac{2}{5}$ for every other type.

Consider candidate z. It is approved by all five voters and belongs to both T_1 and T_2. In each of these two types, the cost $\frac{1}{2}$ can be covered by charging each approving voter $\frac{1}{10}$. For any other type containing z, every voter has balance $\frac{2}{5}$, and again a charge of $\frac{1}{10}$ per approving voter suffices. Hence, $\max_{r\in[d]} \eta_r(z) = \frac{1}{10}$.

By contrast, candidates x and y are approved only by voters 1 and 2, and $x \in T_1$, $y \in T_2$. Thus, the minimum per-voter price needed to buy x or y is $\eta_1(x) = \eta_2(y) = \frac{1}{5}$. Therefore both x and y have price at least $\frac{1}{5}$, while z has price $\frac{1}{10}$.

Every dummy candidate is approved by no voter. Hence any dummy candidate that belongs to at least one type is not affordable, whereas any dummy candidate belonging to no type does not change any voter's balance for any type and does not affect the argument. Thus, τ-MES selects z before selecting either x or y.

After z is selected, each voter has paid $\frac{1}{10}$ from her type-1 balance and $\frac{1}{10}$ from her type-2 balance. Therefore voters 1 and 2 each have remaining type-1 balance $\frac{1}{5} - \frac{1}{10} = \frac{1}{10}$. So, collectively, they have $\frac{1}{5}$ remaining balance for type 1. Since $\mathrm{cost}(x) = \frac{2}{5}$, and voters 1 and 2 are the only ones who approve x, candidate x is no longer affordable. The same argument shows that voters 1 and 2 do not have enough type 2 budget, and thus y is also no longer affordable.

No other candidate approved by voters in S can be selected, since the remaining candidates are dummies approved by no voter. Hence, in the final outcome W returned by τ-MES, the only selected candidate approved by any voter in S is z. Therefore $\left|W \cap \bigcup_{i\in S} A_i\right| = 1 < 2$. Since S is 2-deserving, as witnessed by $X = \{x, y\}$, this means W violates τ-PJR, concluding the proof. ☐

Another well-known and widely studied rule in the PB literature is Phragmén's rule, which has been shown to satisfy PJR, but not EJR, in standard PB [Los et al., 2022]. A natural question is whether an appropriate variant of Phragmén's rule can also guarantee τ-PJR for τ-PB. Therefore, we consider the following variant of Phragmén's rule with type constraints.

Phragmén's rule with type constraints (τ-Phragmén's rule). For each type $t \in [d]$, each voter earns type-t currency continuously at rate $\frac{b_t}{n}$. Initially, all balances are 0, and the rule constructs an outcome W, starting from $W = \emptyset$. For each voter $i \in N$ and each type $t \in [d]$, let $x_i^t(\tau)$ denote the amount of type-t currency held by voter i at time τ. For each candidate $c \in C$, let $\tau(c) = \{t : c \in T_t\}$ be the set of types containing c, and let $N(c) = \{i \in N : c \in A_i\}$.

An unselected candidate $c \in C \setminus W$ is affordable at time τ if $\sum_{i\in N(c)} x_i^t(\tau) \geq \mathrm{cost}(c)$ for every type $t \in \tau(c)$. We say that c is eligible with respect to the current outcome W if $\mathrm{cost}(W \cap T_t) + \mathrm{cost}(c) \leq b_t$ for every type $t \in \tau(c)$. At the earliest time when some unselected candidate is both affordable and eligible, the rule selects one such candidate, breaking ties according to a fixed tie-breaking order.

After selecting c, the voters in $N(c)$ pay for c separately in each type $t \in \tau(c)$. For every type $t \in \tau(c)$, choose a unique threshold $\alpha_t(c) \geq 0$ such that $\sum_{i\in N(c)} \max\{0, x_i^t(\tau) - \alpha_t(c)\} = \mathrm{cost}(c)$. Each voter $i \in N(c)$ then pays $p_i^t(c) = \max\{0, x_i^t(\tau) - \alpha_t(c)\}$ units of type-t currency. After the payment, we update $x_i^t(\tau) \leftarrow x_i^t(\tau) - p_i^t(c)$ for every $i \in N(c)$ and every $t \in \tau(c)$. The balances of voters not approving c remain unchanged. The process stops when no unselected candidate is both affordable and eligible.

Proposition 6. *There exists an instance of τ-PB with laminar type constraints for which τ-Phragmén's rule does not satisfy τ-PJR.*

Proof. Consider an instance with five voters $N = \{1, 2, 3, 4, 5\}$ and three types $[d] = \{1, 2, 3\}$ with $T_1 = \{a, b, c, d, e\}$, $T_2 = \{a, d\}$, $T_3 = \{c, e\}$. Thus, $T_2, T_3 \subseteq T_1$, $T_2 \cap T_3 = \emptyset$, and $\mathcal{H}$ is laminar. The candidate costs are: $\mathrm{cost}(a) = \mathrm{cost}(c) = \frac{2}{5}$, $\mathrm{cost}(d) = \mathrm{cost}(e) = \frac{1}{2}$, $\mathrm{cost}(b) = \frac{6}{5}$. Let the type budgets be $b_1 = 2$, $b_2 = b_3 = 1$. The voters' approval sets are $A_1 = A_2 = \{a, b, c\}$, $A_3 = A_4 = A_5 = \{b, d, e\}$.

We first show that the group $S = \{1, 2\}$ is 2-deserving. Let $X = \{a, c\}$. Then $X \subseteq \bigcap_{i \in S} A_i$ and $|X| = 2$. Note that $\mathrm{cost}(X \cap T_1) = \mathrm{cost}(a) + \mathrm{cost}(c) = \frac{4}{5} \le b_1 \cdot \frac{|S|}{n}$, $\mathrm{cost}(X \cap T_2) = \mathrm{cost}(a) = \frac{2}{5} \le b_2 \cdot \frac{|S|}{n}$, $\mathrm{cost}(X \cap T_3) = \mathrm{cost}(c) = \frac{2}{5} \le b_3 \cdot \frac{|S|}{n}$. Hence, the group S is 2-deserving.

We now run τ-Phragmén's rule. Each voter earns type-1 currency at rate $\frac{2}{5}$, and type-2 and type-3 currency at rate $\frac{1}{5}$.

At time τ, candidate b is affordable exactly when the five approving voters have jointly accumulated at least $\mathrm{cost}(b) = \frac{6}{5}$ units of type-1 currency. Since they jointly earn type-1 currency at rate 2, candidate b becomes affordable at time $\frac{3}{5}$.

By contrast, candidate a is approved by two voters and belongs to types 1 and 2. Hence it becomes affordable exactly when its supporters have accumulated at least $\frac{2}{5}$ units in each of these two types. The type-1 requirement is met at time $\frac{1}{2}$, while the type-2 requirement is met at time 1. Therefore a becomes affordable at time 1. The same calculation shows that c also becomes affordable at time 1.

Similarly, candidate d is approved by three voters and belongs to types 1 and 2. Its type-1 requirement is met at time $\frac{5}{12}$, while its type-2 requirement is met at time $\frac{5}{6}$. Thus d becomes affordable at time $\frac{5}{6}$, and by symmetry so does e.

Since $\frac{3}{5} < \frac{5}{6} < 1$, we conclude that b is the unique first selected candidate.

At time $\frac{3}{5}$, each voter has accumulated $\frac{6}{25}$ units of type-1 currency and $\frac{3}{25}$ units of each of types 2 and 3. The selection of b exhausts all type-1 currency and leaves all type-2 and type-3 balances unchanged. Thus, immediately after selecting b, every voter has balance 0 in type 1, $\frac{3}{25}$ in type 2, $\frac{3}{25}$ in type 3.

We next determine the second selected candidate.

For candidate a, its supporters need an additional $\frac{2}{5}$ units in type 1, which takes $\frac{1}{2}$ units of time, and an additional $\frac{2}{5} - \frac{6}{25} = \frac{4}{25}$ units in type 2, which takes $\frac{2}{5}$ units of time. Hence a becomes affordable again after an additional time of $\max\{\frac{1}{2}, \frac{2}{5}\} = \frac{1}{2}$. The same holds for c.

For candidate d, its supporters need an additional $\frac{1}{2}$ units in type 1, which takes $\frac{5}{12}$ units of time, and an additional $\frac{1}{2} - \frac{9}{25} = \frac{7}{50}$ units in type 2, which takes $\frac{7}{30}$ units of time. Hence d becomes affordable again after an additional time of $\max\{\frac{5}{12}, \frac{7}{30}\} = \frac{5}{12}$. The same holds for e.

Therefore the second selected candidate is either d or e, since $\frac{5}{12} < \frac{1}{2}$.

After selecting either d or e, the total type-1 cost of the selected candidates is $\frac{6}{5} + \frac{1}{2} = \frac{17}{10}$., and the remaining type-1 budget is only $2 - \frac{17}{10} = \frac{3}{10}$. But every remaining candidate has cost at least $\frac{2}{5}$.

Since every remaining candidate belongs to T_1, no remaining candidate is eligible. Hence the rule terminates with $W = \{b, d\}$ or $W = \{b, e\}$. Finally, $U = \bigcup_{i \in S} A_i = \{a, b, c\}$, so in either case $|W \cap U| = 1 < 2$. Thus, the group S are represented by fewer than 2 approved candidates, and the outcome violates τ-PJR.

Therefore τ-Phragmén's rule does not satisfy τ-PJR even when $\mathcal{H}$ is laminar. ☐

We next turn to the question of whether τ-MES can satisfy stronger proportionality axioms under a more restrictive class of constraint structures.

4.3 Multiwinner Voting with Nested Types: τ-MES Satisfies τ-EJR

In this section, we show that τ-MES satisfies the stronger proportionality axiom τ-EJR for multiwinner voting with nested type constraints. We also show that both restrictions are essential, as the guarantee fails if either candidates have heterogeneous costs or the type constraints are not nested.

We say that a τ-PB instance is a multiwinner voting instance with nested type constraints if every candidate has unit cost, there exists a type $t^\star \in [d]$ such that $T_{t^\star} = C$ and $b_{t^\star} = k$, and the constraint structure $\mathcal{H}$ is nested, that is, for every $t, t' \in [d]$, either $T_t \subseteq T_{t'}$ or $T_{t'} \subseteq T_t$.

Theorem 4. *In multiwinner voting with nested type constraints, τ-MES satisfies τ-EJR.*

Proof. Assume, for the sake of a contradiction, that the outcome W violates τ-EJR. Then there exists an integer $1 \leq \ell \leq k$ and a set of voters $S \subseteq N$ that deserve ℓ candidates such that $|A_i \cap W| \leq \ell - 1$ for every $i \in S$. Let $X \subseteq \bigcap_{i \in S} A_i$ be a witnessing set with $|X| \geq \ell$ and $|X_t| \leq b_t \cdot \frac{|S|}{n}$ for every $t \in [d]$. Since all candidates have unit cost, we may discard extra candidates and assume that $|X| = \ell$.

Because the types are nested, we may relabel them so that $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_d$, with $T_1 = C$. For each $t \in [d]$, let $X_t = X \cap T_t$, and set $X_{d+1} = \emptyset$. Define $Y_t = X_t \setminus X_{t+1}$ for $t \in [d]$. Then $(Y_1, \ldots, Y_d)$ forms a partition of X, and every candidate in Y_h belongs exactly to $T_1, \ldots, T_h$.

Since every voter in S approves all candidates in X and $|X| = \ell$, the assumption that $|A_i \cap W| \leq \ell - 1$ for all $i \in S$ implies that $X \setminus W \neq \emptyset$. Let h be the smallest index such that $Y_h \setminus W \neq \emptyset$, and fix some candidate $x \in Y_h \setminus W$. By the choice of h, we have $Y_1 \cup \cdots \cup Y_{h-1} \subseteq W$.

In particular, for every $t \leq h$, all candidates in $Y_1 \cup \cdots \cup Y_{t-1}$ are selected. Since $|Y_1| + \cdots + |Y_{t-1}| = \ell - |X_t|$, every voter in S already approves at least $\ell - |X_t|$ selected candidates outside T_t.

We now prove the key claim.

Claim 2. *As long as x remains unselected, every voter $i \in S$ has remaining balance at least $\frac{1}{|S|}$ in every type $t \leq h$.*

Proof of Claim. Suppose not, and let r be the first round after which there exists a voter $j \in S$ and an index $t \leq h$ such that $p_j^{(t)} < \frac{1}{|S|}$. By the choice of r, before round r every voter in S has balance at least $\frac{1}{|S|}$ in each of the types $1, \ldots, h$.

Since $x \in Y_h \subseteq \bigcap_{i \in S} A_i$ and x belongs to $T_1, \ldots, T_h$, the voters in S alone can afford x before round r by paying $\frac{1}{|S|}$ each in every relevant type. Hence the price of x is at most $\frac{1}{|S|}$ at that moment. Since the algorithm selects an affordable candidate of minimum price, the candidate selected in round r also has price at most $\frac{1}{|S|}$.

Therefore, in every round up to and including round r, voter j pays at most $\frac{1}{|S|}$ in type t whenever she contributes to a selected approved candidate belonging to T_t. Let q be the number of selected candidates in $A_j \cap T_t$ by the end of round r. Then voter j has spent at most $\frac{q}{|S|}$ in type t.

On the other hand, after round r we have $p_j^{(t)} < \frac{1}{|S|}$, so voter j has spent more than $\frac{b_t}{n} - \frac{1}{|S|}$ in type t. Hence $\frac{q}{|S|} > \frac{b_t}{n} - \frac{1}{|S|}$, which implies $q > b_t \cdot \frac{|S|}{n} - 1$. Since q is an integer and $|X_t| \leq b_t \cdot \frac{|S|}{n}$, it follows that $q \geq |X_t|$.

But voter j also approves all $\ell - |X_t|$ already selected candidates in $Y_1 \cup \cdots \cup Y_{t-1}$. Therefore $|A_j \cap W| \geq (\ell - |X_t|) + q \geq \ell$, contradicting the assumption that $|A_j \cap W| \leq \ell - 1$. ☐

Since $x \notin W$, the candidate x remains unselected until the algorithm terminates. By Claim 2, every voter in S still has balance at least $\frac{1}{|S|}$ in each of the types $1, \ldots, h$ at termination. As x is approved by all voters in S and belongs exactly to these types, the voters in S can still afford x at termination. This contradicts the stopping condition of the algorithm.

Hence our assumption was false, and W satisfies τ-EJR. ☐

This result is tight in the sense that τ-MES no longer satisfies τ-EJR if we generalize either to laminar type constraints or to heterogeneous costs, as the following result shows.

Proposition 7. *There exists an instance in each of the following two settings for which τ-MES does not return an outcome satisfying τ-EJR: (1) multiwinner voting with laminar type constraints, and (2) τ-PB with nested type constraints.*

Proof. **(1) Failure under multiwinner voting with laminar types.**

Consider a multi-winner instance with voters $N = \{1, 2, 3, 4\}$ and candidates $C = \{a, b, c, e\}$. Let the type sets be $T_M = C$, $T_L = \{a, b\}$, $T_R = \{c, e\}$, with budgets $b_M = 4$, $b_L = 2$, $b_R = 2$. The family $\mathcal{H} = \{T_M, T_L, T_R\}$ is laminar, but it is not nested, since T_L and T_R are disjoint. The approval sets are $A_1 = \{a, b, c\}$, $A_2 = \{b, e\}$, $A_3 = \{a, c, e\}$, $A_4 = \{b, e\}$.

Let $S = \{1, 3\}$. Consider $X = \{a, c\} \subseteq A_1 \cap A_3$. Notice that we have $|X| = 2$, $|X \cap T_M| = 2 \leq b_M \cdot \frac{|S|}{|N|}$, $|X \cap T_L| = 1 \leq b_L \cdot \frac{|S|}{|N|}$, $|X \cap T_R| = 1 \leq b_R \cdot \frac{|S|}{|N|}$. Hence, S is 2-deserving.

We now run τ-MES. Initially, each voter has balances 1 in type M, $\frac{1}{2}$ in type L, and $\frac{1}{2}$ in type R. The price needed to buy candidate b is $\frac{1}{3}$. The price needed to buy candidate e is also $\frac{1}{3}$. The price needed to buy candidate a or c is $\frac{1}{2}$. Hence τ-MES first selects either b or e.

By symmetry, suppose b is selected first. Then voters $1, 2, 4$ each pay $\frac{1}{3}$ in types M and L. Candidate a is no longer affordable, since its approvers 1 and 3 have total remaining type-L balance of $\frac{1}{6} + \frac{1}{2} = \frac{2}{3} < 1$. Candidate e remains affordable with price $\frac{1}{3}$, while c has price $\frac{1}{2}$. Thus e is selected next. After selecting e, voters $2, 3, 4$ each pay $\frac{1}{3}$ in types M and R. Now c is no longer affordable, since its approvers 1 and 3 have total remaining type-R balance $\frac{1}{2} + \frac{1}{6} = \frac{2}{3} < 1$. No unselected candidate is affordable, so τ-MES terminates with $W = \{b, e\}$. The case where e is selected first is symmetric and leads to the same outcome.

However, voter 1 approves only one selected candidate, and voter 3 approves only one selected candidate. Thus no voter in S receives 2 approved candidates, even though S is 2-deserving. Therefore τ-MES fails τ-EJR even under laminar type constraints when the type family is not nested.

(2) Failure under τ-PB with nested types.

Consider the instance with voters $N = \{1, 2, 3\}$ and candidates $C = \{a, b, c, d, e\}$. Let $T_3 = \{a, d\} \subset T_2 = \{a, b, d\} \subset T_1 = \{a, b, c, d, e\}$, so the type structure is nested. Let the type budgets be $b_1 = \frac{11}{2}, b_2 = 2$, $b_3 = 2$. Let the costs of candidates be $\mathrm{cost}(a) = \mathrm{cost}(b) = \mathrm{cost}(c) = 1$, $\mathrm{cost}(d) = \frac{4}{3}$, $\mathrm{cost}(e) = \frac{7}{3}$. Finally, the approval sets are $A_1 = \{a, b, d, e\}$, $A_2 = \{a\}$, $A_3 = \{c, d, e\}$.

Let $S = \{1, 3\}$. Then S is 2-deserving, witnessed by $X = \{d, e\} \subseteq A_1 \cap A_3$.

We now run τ-MES. Initially the voters' balances are $\left(\frac{11}{6}, \frac{2}{3}, \frac{2}{3}\right)$ for types 1, 2, 3, respectively.

At the first step, candidate a has price $\frac{1}{2}$, candidate d has price $\frac{2}{3}$, candidate c has price 1, and candidate e has price $\frac{7}{6}$. Candidate b is not affordable, since its approver only has balance $\frac{2}{3}$ in type 2. Hence a is the candidate chosen first.

After selecting a, voters 1 and 2 each pay $\frac{1}{2}$ in every relevant type. Now b is still not affordable, and d is not affordable because its approvers $\{1, 3\}$ have total type-2 balance $\frac{1}{6} + \frac{2}{3} = \frac{5}{6} < \frac{4}{3}$. Candidate c is affordable with price 1, while candidate e is affordable with price $\frac{7}{6}$. Thus c is the unique second choice.

After selecting c, voter 3 pays 1 in type 1, so the remaining type-1 balances of voters 1 and 3 are $\frac{4}{3}$ and $\frac{5}{6}$. Hence e is no longer affordable, since $\frac{4}{3} + \frac{5}{6} = \frac{13}{6} < \frac{7}{3}$. Candidates b and d are still not affordable. Therefore the algorithm terminates with $W = \{a, c\}$.

Finally, $|A_1 \cap W| = 1$ and $|A_3 \cap W| = 1$, so no voter in S receives 2 approved candidates, even though S is 2-deserving. Thus τ-MES fails τ-EJR. $\square$

We next establish that τ-Phragmén's rule satisfies τ-PJR for multiwinner voting with nested types. We also show that this guarantee cannot be strengthened to τ-EJR.

Theorem 5. *In multiwinner voting with nested types, τ-Phragmén's rule satisfies τ-PJR.*

Proof. Let W be the committee returned by τ-Phragmén, and let $\widehat{W} \subseteq W$ be the set of candidates selected by time 1. Also note that by this time the total type-t currency given to the voters is $\frac{b_t}{n} \cdot n = b_t$.

Suppose, for a contradiction, that W violates τ-PJR. Then there exist $\ell \in [k]$, a group $S \subseteq N$, and a witness set $X \subseteq \bigcap_{i \in S} A_i$ with $|X| = \ell$ and $|X_t| \leq b_t \cdot \frac{|S|}{n}$ for every type t, such that $|W \cap U| < \ell$, where $U = \bigcup_{i \in S} A_i$. Since $\widehat{W} \subseteq W$, we also have $|\widehat{W} \cap U| < \ell$.

Because the types are nested, relabel them so that $T_1 = C \supseteq T_2 \supseteq \cdots \supseteq T_d$. Let $X_t = X \cap T_t$, set $X_{d+1} = \emptyset$, and define $Y_t = X_t \setminus X_{t+1}$. Then $X = Y_1 \cup \cdots \cup Y_d$, and every candidate in Y_h belongs exactly to $T_1, \ldots, T_h$.

Since $|X| = \ell$ but $|\widehat{W} \cap U| < \ell$, some candidate in X is not in $\widehat{W}$. Let h be the smallest index with $Y_h \setminus \widehat{W} \neq \emptyset$, and fix $x \in Y_h \setminus \widehat{W}$. By minimality of h, all candidates in $Y_1 \cup \cdots \cup Y_{h-1}$ belong to $\widehat{W}$.

Fix any $t \leq h$. The set $Y_1 \cup \cdots \cup Y_{t-1} = X \setminus X_t$ has size $\ell - |X_t|$, lies outside T_t, and is contained in $\widehat{W} \cap U$. Therefore $|\widehat{W} \cap U \cap T_t| = |\widehat{W} \cap U| - |(\widehat{W} \cap U) \setminus T_t| = |\widehat{W} \cap U| - (\ell - |X_t|) < |X_t|$, and hence $|\widehat{W} \cap U \cap T_t| \leq |X_t| - 1$.

By time 1, voters in S have earned $|S| \cdot \frac{b_t}{n} \geq |X_t|$ units of type-t currency. They can spend type-t currency only on candidates in $\widehat{W} \cap U \cap T_t$, and each such candidate costs at most one unit of type-t currency from S. Thus, at time 1, voters in S still hold at least one unit of type-t currency. Since this holds for every $t \leq h$ and x belongs exactly to $T_1, \ldots, T_h$, the voters in S can jointly pay for x at time 1.

Moreover, for every $t \leq h$, because voters in S still hold positive type-t currency at time 1, we know the total amount of type-t currency spent by time 1 is strictly less than b_t. Hence $|\widehat{W}_t| \leq b_t - 1$, and adding x would not violate any type budget. Thus x is both affordable and eligible at time 1, contradicting the definition of $\widehat{W}$.

Therefore $|\widehat{W} \cap U| \geq \ell$. Since $\widehat{W} \subseteq W$, we get $|W \cap U| \geq \ell$, contradicting the assumed violation of τ-PJR. Hence, τ-Phragmén satisfies τ-PJR for multiwinner voting with nested types. ☐

However, this result cannot be strengthened to τ-EJR. This is true because in the special case of multiwinner voting (with no additional type constraints), τ-EJR coincides with EJR and τ-Phragmén coincides with Sequential Phragmén, and it is known that Sequential Phragmén does not satisfy EJR [Brill et al., 2024].

4.4 Arbitrary Types: Existence and Complexity

With Proposition 5, we established that τ-MES cannot guarantee τ-PJR when the constraint structure is not laminar. Given that none of our studied rules satisfy τ-PJR for arbitrary type constraints, it is reasonable to first ask whether an outcome satisfying τ-PJR is guaranteed to exist in this case. We will now answer this question in the affirmative, and moreover, show that a property stronger than τ-EJR is guaranteed to exist.

In the PB setting without candidate types, *full justified representation* strengthens EJR by weakening the formulation of cohesiveness [Peters et al., 2021]. We can analogously weaken the notion of deserving candidates in our setting, and thus broaden the groups of voters that are entitled to representation. To that end, we say a group of voters $S \subseteq N$ *weakly deserves* ℓ candidates in *τ-PB* if there exists a set $X \subseteq C$ such that, for every voter $i \in S$, we have $|A_i \cap X| \geq \ell$, and for every type $t \in [d]$, we have $\mathrm{cost}(X_t) \leq b_t \cdot \frac{|S|}{n}$. In other words, a group weakly deserves ℓ candidates if there exists a set of candidates that gives every voter in the group utility at least ℓ, while staying within the group's proportional share of every type budget. We can now define τ-FJR, which guarantees representation to all weakly deserving groups, and thus strengthens EJR.

Definition 5 (Full Justified Representation with Type Constraints (τ-FJR))**.** A feasible outcome W satisfies τ-FJR if for every integer $\ell \geq 1$ and every group of voters $S \subseteq N$ that weakly deserves ℓ candidates in *τ-PB*, there exists a voter $i \in S$ such that $|A_i \cap W| \geq \ell$.

In the PB setting with only a single budget constraint, FJR is known to exist and is satisfied by the Greedy Cohesive Rule (GCR). The extension of GCR to our setting is trivial and proceeds as follows: start with an empty outcome $W = \emptyset$ and label all voters as active. Iteratively, search for a group S of active voters that weakly deserves ℓ candidates, and denote the group's certificate candidate set by X. Choose such a group for maximum ℓ, breaking ties in favor of smaller $\mathrm{cost}(X)$. Add to W the group's certificate X and label all voters in S as inactive. It is relatively straightforward to show that this procedure guarantees τ-FJR, yielding the following statement.

Proposition 8. *In τ-PB with arbitrary type constraints, a feasible outcome satisfying τ-FJR is guaranteed to exist.*

Proof. We prove the statement constructively by showing the extension of GCR given above is guaranteed to satisfy FJR. First, we note that GCR always returns a feasible outcome. In each iteration, GCR deactivates a group of voters S and adds a set of candidates X. Since X is the certificate that S weakly deserves some number of candidates, it must be that $\mathrm{cost}(X_t) \leq b_t \cdot |S|/n$ for every $t \in [d]$. For each type, the cost of the eventual outcome W is upper bounded by the sum of the certificates X in each round. Thus, when all voters are marked inactive and the algorithm terminates, it holds for each $t \in [d]$ that $\mathrm{cost}(W) \leq b_t \cdot n/n = b_t$.

Now, to see that GCR satisfies τ-FJR, consider an arbitrary group of voters $S \subseteq N$ that weakly deserves $\ell \geq 1$ candidates and let X denote their certificate set. Let $i \in S$ be the first voter in S who is marked inactive during the execution of GCR. Before this point, every voter in S is active, and since S weakly deserves ℓ candidates, the group the algorithm sets inactive during the next iteration must weakly deserve $\ell' \geq \ell$ candidates. Thus $|A_i \cap W| \geq \ell' \geq \ell$, and W satisfies τ-FJR. ☐

The algorithm used to constructively prove the existence of FJR (resp. τ-FJR) takes exponential time. In the multiwinner voting setting, it remains open whether a polynomial time rule for FJR exists. By contrast, we show that in the presence of arbitrary type constraints, no polynomial time rule can guarantee even τ-PJR in multiwinner voting with a single voter, unless P = NP.

Theorem 6. *Unless P = NP, no polynomial-time rule satisfies τ-PJR in τ-PB with arbitrary type constraints, even in the special case of unit costs and $n = 1$.*

Proof. We reduce from INDEPENDENT SET. Let (G, K) be an instance, where $G = (V, E)$. We construct a single-voter, unit-cost τ-PB instance $\mathcal{I} = \big(N, C, \mathrm{cost}, (A_i)_{i \in N}, \mathcal{H}, (b_t)_{t \in [d]}\big)$ as follows.

Let $N = \{1\}$ consist of a single voter. For each vertex $v \in V$, create a candidate c_v, and set $C = \{c_v : v \in V\}$. All candidates have unit cost, and the unique voter approves all candidates, so $A_1 = C$. For each edge $e = \{u, v\} \in E$, create a type t_e with $T_{t_e} = \{c_u, c_v\}$ and budget $b_{t_e} = 1$. We also create a global type $t^\star$ with $T_{t^\star} = C$ and $b_{t^\star} = K$.

A set $W \subseteq C$ is feasible if and only if $|W| \leq K$ and, for every edge $e = \{u, v\}$, at most one of c_u, c_v belongs to W. Thus feasible outcomes correspond exactly to independent sets in G of size at most K.

Let $\alpha(G)$ be the size of a maximum independent set in G. We claim that every feasible outcome satisfying τ-PJR has size exactly $\min\{K, \alpha(G)\}$. Since all feasible outcomes correspond to independent sets of size at most K, every feasible outcome has size at most $\min\{K, \alpha(G)\}$.

Conversely, consider the singleton voter group $S = N = \{1\}$, and this group is entitled to the full budget of every type. Hence, S is ℓ-deserving if and only if there exists a feasible set of ℓ candidates, which holds if and only if G has an independent set of size ℓ with $\ell \leq K$. In particular, S is $\min\{K, \alpha(G)\}$-deserving. Because the unique voter approves every candidate, any outcome satisfying τ-PJR must give this voter utility at least $\min\{K, \alpha(G)\}$, and therefore must have size at least $\min\{K, \alpha(G)\}$.

Thus, we conclude that every feasible outcome satisfying τ-PJR has size exactly $\min\{K, \alpha(G)\}$. As a result, a voting rule which guarantees τ-PJR returns an outcome of size K if and only if G has an independent set of size at least K. Hence, unless P = NP, no polynomial-time voting rule can guarantee τ-PJR for τ-PB with arbitrary type constraints, even in the special case of unit costs and a single voter. ☐

Theorem 6 affirms that our positive algorithmic results could not have been extended to arbitrary type constraints with a different rule. In the construction used for the proof of Theorem 6, each type comprises exactly two candidates, and each candidate may belong to at most m types. One possible approach to obtaining proportionality guarantees in polynomial time would be to place restrictions on type constraint size and/or number of types per candidate.

5 Experiments: Participatory Budgeting in Amsterdam and Warsaw

In this section, we evaluate our voting rules on real-world participatory budgeting data collected by Faliszewski et al. [2023] from elections held in the Netherlands and Poland.

Instance	Voters	Projects	Budget	#types	Min type budget	Max type budget	Avg. type budget
KW	426	52	€250,000	6	€33,000	€54,000	€41,667
BB	219	24	€250,000	3	€53,000	€115,000	€83,333
CB	8,520	85	€500,000	3	€100,000	€250,000	€166,667
GS	5,510	97	€400,000	3	€100,000	€200,000	€133,333
OO	2,324	24	€300,000	4	€57,312	€90,610	€75,000
IZ	2,037	25	€300,000	4	€56,916	€91,354	€75,000

Table 1: Basic statistics of the six Amsterdam instances used in the experiments.

Instance	Type	Budget
KW	Armoede	€52,000
KW	Eenzaamheid	€37,000
KW	Groenonderhoud straten & pleinen	€35,000
KW	Jeugdactiviteiten	€54,000
KW	Rattenpreventie	€39,000
KW	Sportactiviteiten	€33,000
BB	Jeugd	€82,000
BB	Groen	€53,000
BB	Ontmoeting	€115,000
CB	Oostelijke Eilanden	€150,000
CB	Haarlemmerhouttuinen	€100,000
CB	Other	€250,000
GS	Straten pleinen en parken	€200,000
GS	Gezondheid cultuur en kansen voor iedereen	€100,000
GS	Samen dingen doen	€100,000
OO	Meer groen in de buurt	€90,610
OO	Kindvriendelijke buurt	€57,312
OO	Kinder- en jongerenactiviteiten	€71,364
OO	Klimaat en duurzaamheid	€80,715
IZ	Meer groen in de buurt	€91,354
IZ	Plekken voor jongeren	€71,896
IZ	Kinder- en jongerenactiviteiten	€56,916
IZ	Minder zwerfvuil/grof afval	€79,834

Table 2: Type budgets in the six Amsterdam instances.

5.1 Participatory budgeting in Amsterdam with type budgets

We first focus on six district-level approval-based PB elections from Amsterdam that contain explicit type-budget constraints: KLEINE WERELD (KW), BLOEMENBUURT (BB), CENTRUM BEGROOT (CB), GEUZENVELD SLOTERMEER (GS), OUD OOST (OO), and IJBURG ZEEBURGEREILAND (IZ).

The six instances vary considerably in scale, ranging from BB, with 219 voters and 24 projects, to CB, with 8,520 voters and 85 projects. Their overall budgets lie between €250,000 and €500,000. In five of these instances, category budgets are specified alongside the overall budget, with each category budget imposing an upper bound on the total cost of selected projects in the corresponding category. In CENTRUM BEGROOT, by contrast, the data specifies neighborhood budgets, each imposing an upper bound on the total cost of selected projects in the corresponding neighborhood. Specifically, KW contains 426 voters, 52 projects, and six category budgets under a total budget of €250,000. BB contains 219 voters, 24 projects, and three category budgets under €250,000. CB contains 8,520 voters, 85 projects, and three neighborhood budgets under €500,000. GS contains 5,510 voters, 97 projects, and three category budgets under €400,000. OO contains 2,324 voters, 24 projects, and four category budgets under €300,000. IZ contains 2,037 voters, 25 projects, and four category budgets under €300,000. Table 1 reports their basic statistics, while Table 2 lists the type budgets in each instance.

Instance	#Type Constraints*	Sequential PAV	Phragmén	MES	τ-Phragmén	τ-MES
KW	6	✗	✗	✔	✔	✔
BB	3	✗	✗	✗	✔	✔
CB	3	✗	✗	✗	✔	✔
GS	3	✗	✗	✔	✔	✔
OO	4	✗	✗	✗	✔	✔
IZ	4	✗	✗	✗	✔	✔

Table 3: Whether outcomes selected by the rules satisfy the specified type budgets. * The overall budget constraint is not counted.

Instance	Total Welfare ↑				Welfare / €1,000 ↑			
	τ-Seq PAV	τ-Phragmén	τ-MES	τ-MES (Completion)	τ-Seq PAV	τ-Phragmén	τ-MES	τ-MES (Completion)
KW	3,324	**3,729**	2,355	**3,729**	13.9	19.0	**39.4**	19.0
BB	1,397	**1,777**	1,572	**1,777**	8.4	11.8	**14.6**	11.8
CB	60,928	88,413	68,759	**88,943**	126.2	186.0	**322.0**	187.5
GS	10,253	**13,618**	5,888	**13,618**	25.9	34.4	**69.6**	34.4
OO	**22,538**	21,697	17,859	21,697	88.2	95.5	**109.7**	95.5
IZ	7,680	**9,088**	4,728	8,670	28.8	36.6	**51.5**	37.2

Table 4: τ-Seq PAV, τ-Phragmén, τ-MES, and τ-MES (Completion) on all six Amsterdam instances. Within each metric block, the numerically largest entry in each row is highlighted in bold.

Existing rules using the overall budget only. For each instance, we first ignore the type budgets and run three unconstrained approval-based rules using only the overall budget:

- **MES**, the standard Method of Equal Shares **without** type constraints
- **Phragmén's rule**, the standard Phragmén voting rule **without** type constraints
- **Sequential PAV**, the greedy rule that repeatedly selects one project to maximize the Proportional Approval Voting (PAV) score

We then check whether the selected outcome violates the specified category or neighborhood budgets. Table 3 shows that both Phragmén and Sequential PAV violate the specified budgets in all six instances. MES also violates the specified budgets in four of the six cases. By contrast, τ-Phragmén and τ-MES satisfy the type-budget constraints in all instances.

Comparison under type constraints. Having established that unconstrained rules may violate the specified type budgets, we next solve all six Amsterdam instances under explicit type-budget constraints. For each instance, we compare four voting rules:

1. τ-**Seq PAV**, which greedily adds a feasible project maximizing the increase in PAV score
2. τ-**Phragmén's rule**, Phragmén's rule with type constraints
3. τ-**MES**, Algorithm 1
4. τ-**MES (Completion)**, a completion heuristic that first runs τ-MES until it terminates and then continues with τ-Phragmén on the residual instance [5]

In Table 4, we report two evaluation metrics. The first is total welfare, $\sum_{i \in N} u_i(W)$, and the second is the welfare-per-cost ratio, $\frac{\sum_{i \in N} u_i(W)}{\mathrm{cost}(W)}$, which measures the approval utility obtained per unit of budget spent. Across all six instances, τ-MES achieves the highest welfare-per-cost ratio. τ-Phragmén and τ-Seq PAV generally obtain higher total welfare, but at lower efficiency. τ-MES (Completion) is an intermediate alternative, often matching τ-Phragmén and occasionally improving on it slightly.

5.2 Participatory budgeting in Warsaw with overlapping types

We note that the Amsterdam PB instances have simple type constraint structures, where each project is subject to the global budget and exactly one category/neighborhood budget. We additionally perform

---

[5] This approach is common in standard PB. We refer the reader to a detailed discussion by Peters and Skowron [2023].

Year	#Projects	#Category labels	#Beneficiary labels	Multi-category projects	Multi-beneficiary projects	Avg. category labels/project	Avg. beneficiary labels/project
2020	101	9	6	72	85	2.13	3.50
2021	106	10	7	92	101	2.33	3.93
2022	129	10	8	97	114	2.20	3.60
2023	138	10	7	112	130	2.35	4.14
2024	118	9	3	99	83	2.79	1.70

Table 5: Label statistics for the five Warsaw datasets.

Instance	τ-Seq PAV	τ-Phragmén	τ-MES	τ-MES (Completion)
W20-cat	0.83	**1.0**	**1.0**	**1.0**
W20-ben	0.65	**1.0**	**1.0**	**1.0**
W20-comb	0.94	**1.0**	**1.0**	**1.0**
W21-cat	0.99	**1.0**	**1.0**	**1.0**
W21-ben	0.66	**1.0**	**1.0**	**1.0**
W21-comb	0.89	**1.0**	**1.0**	**1.0**
W22-cat	0.92	**1.0**	**1.0**	**1.0**
W22-ben	0.67	**1.0**	**1.0**	**1.0**
W22-comb	0.86	**1.0**	**1.0**	**1.0**
W23-cat	0.93	**1.0**	**1.0**	**1.0**
W23-ben	0.29	**1.0**	**1.0**	**1.0**
W23-comb	0.81	**1.0**	**1.0**	**1.0**
W24-cat	0.94	**1.0**	**1.0**	**1.0**
W24-ben	0.34	**1.0**	**1.0**	**1.0**
W24-comb	0.82	**1.0**	**1.0**	**1.0**
Overall	0.77	**1.0**	**1.0**	**1.0**

Table 6: τ-EJR satisfaction ratios on PB instances in Warsaw. Here, W20-cat denotes the Warsaw 2020 instance with categories as types, W20-ben denotes the Warsaw 2020 instance with beneficiaries as types, and W20-comb denotes the Warsaw 2020 instance with both categories and beneficiaries as types. The other years are defined analogously.

experiments on PB elections in Warsaw from 2020 to 2024, where projects may have multiple types. For each election, we construct three variants using categories, beneficiaries, or both as types.

The resulting type systems are substantially richer than in Amsterdam. The 2020 dataset contains 9 category labels and 6 beneficiary labels, yielding 15 non-global types in the combined variant. The corresponding numbers are 10 and 7 in 2021, 10 and 8 in 2022, 10 and 7 in 2023, and 9 and 3 in 2024. Multi-label projects are common throughout: for example, 72 of the 101 projects in 2020 and 112 of the 138 projects in 2023 belong to multiple categories, while 85 of the 101 projects in 2020 and 130 of the 138 projects in 2023 belong to multiple beneficiary groups. Table 5 summarizes these label statistics.

Since these datasets do not specify type budgets, we calibrate a baseline budget b_t for each non-global type which is equal to the total cost of the official winners of type t. We then sample each non-global type budget from $[0, 2b_t]$ and run the rules.

Overlapping types in Warsaw. For each of the 15 Warsaw instances, we sample 100 budget vectors by drawing every non-global type budget uniformly and independently from $[0, 2b_t]$, where b_t is the baseline budget equal to the total cost of the official winners of type t. For each instance, we evaluate four rules: τ-Seq PAV, τ-Phragmén, τ-MES, and τ-MES (Completion). Tables 6 to 8 report three statistics: the τ-EJR satisfaction ratio, average total welfare, and average welfare-per-cost. Since τ-EJR implies τ-PJR, the same runs also satisfy exact τ-PJR. Across the 1500 sampled budget environments, τ-Phragmén, τ-MES, and τ-MES (Completion) satisfy τ-EJR in every run, whereas τ-Seq PAV does so in only 76.9% of them. In terms of total welfare, τ-MES (Completion) performs best overall while τ-MES achieves by far the highest welfare-per-cost ratio. Although Section 4 shows that our voting rules do not guarantee τ-EJR on all instances with arbitrary type constraints, our experiment is consistent with empirical evidence that they often satisfy proportionality axioms in practice [Boehmer et al., 2026].

Instance	τ-Seq PAV	τ-Phragmén	τ-MES	τ-MES (Completion)
W20-cat	348.6	377.8	233.6	**378.3**
W20-ben	314.7	358.4	250.2	**359.7**
W20-comb	257.5	280.2	143.3	**281.1**
W21-cat	348.9	400.3	241.8	**402.0**
W21-ben	277.4	348.8	224.9	**349.3**
W21-comb	229.4	269.7	147.6	**270.5**
W22-cat	238.3	275.0	156.7	**275.7**
W22-ben	205.0	273.2	166.3	**273.9**
W22-comb	166.5	199.3	94.5	**199.7**
W23-cat	225.7	250.4	104.9	**251.7**
W23-ben	166.8	248.5	174.5	**248.9**
W23-comb	148.7	181.7	79.0	**182.6**
W24-cat	268.0	297.8	176.2	**299.1**
W24-ben	235.6	318.7	243.5	**321.6**
W24-comb	205.9	232.9	118.6	**235.3**
Overall	242.5	287.5	170.4	**288.6**

Table 7: Average welfare, in thousands, on PB instances in Warsaw. Within each row, the best entry is highlighted in bold.

Instance	τ-Seq PAV	τ-Phragmén	τ-MES	τ-MES (Completion)
W20-cat	18.9	20.5	**37.3**	20.5
W20-ben	22.6	26.7	**54.9**	26.7
W20-comb	26.5	30.5	**83.8**	30.5
W21-cat	19.5	23.3	**48.4**	23.3
W21-ben	21.2	28.9	**55.6**	28.8
W21-comb	27.6	35.5	**109.1**	35.7
W22-cat	11.2	13.0	**36.9**	13.1
W22-ben	8.9	12.7	**36.0**	12.8
W22-comb	12.7	16.4	**59.0**	16.4
W23-cat	9.6	11.4	**30.7**	11.4
W23-ben	10.9	16.2	**40.2**	16.1
W23-comb	14.7	18.5	**73.1**	18.8
W24-cat	11.3	12.9	**27.1**	12.9
W24-ben	9.0	12.3	**21.2**	12.4
W24-comb	15.0	17.8	**38.7**	17.9
Overall	16.0	19.8	**50.1**	19.8

Table 8: Average welfare per 1,000 PLN on PB instances in Warsaw. Project costs are measured in PLN. Within each row, the best entry is highlighted in bold.

6 Discussion

We introduced approval-based participatory budgeting with type constraints, where projects may belong to multiple types and each type is associated with its own budget limit. Since existing approval-based voting rules may fail to satisfy standard proportionality axioms even in the committee renewal special case, we introduce type-aware proportionality axioms. We propose type-aware adaptations of the Method of Equal Shares and Phragmén's rule, called τ-MES and τ-Phragmén, and prove positive guarantees for these rules under structured type constraints.

Several directions remain for future work. On the algorithmic side, it is natural to ask whether there exist polynomial-time rules that satisfy τ-EJR beyond multiwinner voting with nested type constraints, or that satisfy τ-PJR in τ-PB under natural non-laminar constraint structures. Another direction is to extend our model to include lower quotas on per-type spending. While our τ-PB framework captures type-aware participatory budgeting through upper spending limits, lower spending requirements are also common in practice and would be worthwhile to study. Our axioms, as defined herein, would require careful extension to meaningfully capture proportionality in this setting.

In multiwinner and PB settings without types, Brill and Peters [2023] introduced the PJR+ and EJR+ axioms, which strengthen PJR and EJR, respectively, and are verifiable in polynomial time. It would also be interesting to see whether an analogous modification can be made to our axioms. Such a modification would require more than a straightforward application of the approach used by Brill and Peters [2023] as it would need to account for each of the types an unselected candidate belongs to.

References

Mohsen Abbasi, Aditya Bhaskara, and Suresh Venkatasubramanian. Fair clustering via equitable group representations. In *Proceedings of the 2021 ACM conference on fairness, accountability, and transparency*, pages 504–514, 2021.

Haris Aziz and Barton E Lee. Proportionally representative participatory budgeting with ordinal preferences. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 5110–5118, 2021.

Haris Aziz and Nisarg Shah. Participatory budgeting: Models and approaches. In *Pathways between social science and computational social science: theories, methods, and interpretations*, pages 215–236. Springer, 2020.

Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, 48(2):461–485, 2017.

Haris Aziz, Edith Elkind, Shenwei Huang, Martin Lackner, Luis Sánchez-Fernández, and Piotr Skowron. On the complexity of extended and proportional justified representation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, 2018a.

Haris Aziz, Barton E Lee, and Nimrod Talmon. Proportionally representative participatory budgeting: Axioms and algorithms. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems*, pages 23–31, 2018b.

Heather Blakey. Participatory budgeting in the UK. *Towards Empowered Participation: Stories and Reflections*, 58:61, 2008.

Niclas Boehmer, Lara Glessen, and Jannik Peters. Understanding the impact of proportionality in approval-based multiwinner elections. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 40, pages 16709–16716, 2026.

Robert Bredereck, Piotr Faliszewski, Ayumi Igarashi, Martin Lackner, and Piotr Skowron. Multiwinner elections with diversity constraints. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, pages 933–940, 2018.

Robert Bredereck, Till Fluschnik, and Andrzej Kaczmarczyk. When votes change and committees should (not). In *Proceedings of the 31st International Joint Conference on Artificial Intelligence, IJCAI 2022*, pages 144–150, 2022.

Markus Brill and Jannik Peters. Robust and verifiable proportionality axioms for multiwinner voting. In *Proceedings of the 24th ACM Conference on Economics and Computation*, page 301, 2023.

Markus Brill, Stefan Forster, Martin Lackner, Jan Maly, and Jannik Peters. Proportionality in approval-based participatory budgeting. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 37, pages 5524–5531, 2023.

Markus Brill, Rupert Freeman, Svante Janson, and Martin Lackner. Phragmén's voting methods and justified representation. *Mathematical Programming*, 203(1):47–76, 2024.

Laurent Bulteau, Noam Hazon, Rutvik Page, Ariel Rosenfeld, and Nimrod Talmon. Justified representation for perpetual voting. *IEEE Access*, 9:96598–96612, 2021.

Yves Cabannes. Participatory budgeting: a significant contribution to participatory democracy. *Environment and Urbanization*, 16(1):27–46, 2004.

Matthew M. Casey and Edith Elkind. Justified representation: From Hare to Droop. *arXiv preprint arXiv:2508.00811*, 2025.

L Elisa Celis, Lingxiao Huang, and Nisheeth K Vishnoi. Multiwinner voting with fairness constraints. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence*, pages 144–151, 2018.

Nikhil Chandak, Shashwat Goel, and Dominik Peters. Proportional aggregation of preferences for sequential decision making. *Journal of Artificial Intelligence Research*, 85, 2026.

Jiehua Chen, Martin Lackner, and Jan Maly. Participatory budgeting with donations and diversity constraints. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 9323–9330, 2022.

Yu Cheng, Zhihao Jiang, Kamesh Munagala, and Kangning Wang. Group fairness in committee selection. *ACM Transactions on Economics and Computation (TEAC)*, 8(4):1–18, 2020.

Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, and Sergei Vassilvitskii. Fair clustering through fairlets. *Advances in neural information processing systems*, 30, 2017.

Julian Chingoma, Umberto Grandi, and Arianna Novaro. Proportionality for constrained public decisions. *arXiv preprint arXiv:2409.02609*, 2024.

Shuang Cui, Kai Han, Shaojie Tang, Feng Li, and Jun Luo. Fairness in streaming submodular maximization subject to a knapsack constraint. In *Proceedings of the 30th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, pages 514–525, 2024.

Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. A new multilayered pcp and the hardness of hypergraph vertex cover. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 595–601, 2003.

Chris Dong and Jannik Peters. Maintaining proportional committees with dynamic candidate sets. In *Forty-second International Conference on Machine Learning*, 2025.

Chris Dong, Fabian Frank, Jannik Peters, and Warut Suksompong. Reconfiguring proportional committees. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 40, pages 16829–16837, 2026.

Carol Ebdon and Aimee L Franklin. Citizen participation in budgeting theory. *Public Administration Review*, 66(3):437–447, 2006.

Edith Elkind, Svetlana Obraztsova, and Nicholas Teh. Temporal fairness in multiwinner voting. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 38, pages 22633–22640, 2024.

Edith Elkind, Svetlana Obraztsova, Jannik Peters, and Nicholas Teh. Verifying proportionality in temporal voting. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 39, pages 13805–13813, 2025.

Roy Fairstein, Reshef Meir, and Kobi Gal. Proportional participatory budgeting with projects interaction. *arXiv preprint arXiv:2106.05360*, 2021.

Piotr Faliszewski, Jarosław Flis, Dominik Peters, Grzegorz Pierczyński, Piotr Skowron, Dariusz Stolicki, Stanisław Szufa, and Nimrod Talmon. Participatory budgeting: Data, tools and analysis. In *32nd International Joint Conference on Artificial Intelligence (IJCAI-23)*, pages 2667–2674. International Joint Conferences on Artificial Intelligence Organization, 2023.

Till Fluschnik, Piotr Skowron, Mervin Triphaus, and Kai Wilker. Fair knapsack. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 1941–1948, 2019.

Mohak Goyal, Sahasrajit Sarmasarkar, and Ashish Goel. A mechanism for participatory budgeting with funding constraints and project interactions. In *International Conference on Web and Internet Economics*, pages 329–347. Springer, 2023.

D Ellis Hershkowitz, Anson Kahng, Dominik Peters, and Ariel D Procaccia. District-fair participatory budgeting. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 5464–5471, 2021.

Pallavi Jain, Krzysztof Sornat, Nimrod Talmon, and Meirav Zehavi. Participatory budgeting with project groups. In *30th International Joint Conference on Artificial Intelligence, IJCAI 2021*, pages 276–282. International Joint Conferences on Artificial Intelligence, 2021.

Yusuf Hakan Kalayci, Jiasen Liu, and David Kempe. Full proportional justified representation. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems*, pages 1070–1078, 2025.

Hans Kellerer, Ulrich Pferschy, and David Pisinger. Multidimensional knapsack problems. In *Knapsack Problems*, pages 235–283. Springer, 2004.

Sonja Kraiczy and Edith Elkind. An adaptive and verifiably proportional method for participatory budgeting. In *International Conference on Web and Internet Economics*, pages 438–455. Springer, 2023.

Martin Lackner and Piotr Skowron. *Multi-winner voting with approval preferences*. Springer, 2023.

Maaike Los, Zoé Christoff, and Davide Grossi. Proportional budget allocations: Towards a systematization. *arXiv preprint arXiv:2203.12324*, 2022.

Tomáš Masařík, Grzegorz Pierczyński, and Piotr Skowron. A generalised theory of proportionality in collective decision making. In *Proceedings of the 25th ACM Conference on Economics and Computation*, pages 734–754, 2024.

Tyanai Masiya, Adrino Mazenda, and Thandokazi NS Gwabeni. Participatory budgeting in a South African local municipality. *International Journal of Management Practice*, 14(3):325–342, 2021.

Ivan-Aleksandar Mavrov, Kamesh Munagala, and Yiheng Shen. Fair multiwinner elections with allocation constraints. In *Proceedings of the 24th ACM Conference on Economics and Computation*, pages 964–990, 2023.

Nima Motamed, Arie Soeteman, Simon Rey, and Ulle Endriss. Participatory budgeting with multiple resources. In *Conference Proceedings of EUMAS*, pages 330–347, 2022.

Dominik Peters and Piotr Skowron. Proportionality and the limits of welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 793–794, 2020.

Dominik Peters and Piotr Skowron. Completion of the method of equal shares. `https://equalshares.net`, 2023. Online; accessed 6-May-2026.

Dominik Peters, Grzegorz Pierczyński, and Piotr Skowron. Proportional participatory budgeting with additive utilities. *Advances in Neural Information Processing Systems*, 34:12726–12737, 2021.

Bradley Phillips, Edith Elkind, Nicholas Teh, and Tomasz Was. Strengthening proportionality in temporal voting. In *Proceedings of AAMAS*, 2026.

Simon Rey, Ulle Endriss, and Ronald De Haan. A general framework for participatory budgeting with additional constraints. *Social Choice and Welfare*, 64(1):5–41, 2025a.

Simon Rey, Felicia Schmidt, and Jan Maly. The (computational) social choice take on indivisible participatory budgeting. *arXiv preprint arXiv:2303.00621*, 2025b.

Anja Röcke. *Framing citizen participation: Participatory budgeting in France, Germany and the United Kingdom*. Springer, 2014.

Irene S Rubin. *The politics of public budgeting: Getting and spending, borrowing and balancing*. CQ Press, 2019.

Luis Sánchez-Fernández, Edith Elkind, Martin Lackner, Norberto Fernández García, Jesús A Fisteus, Pablo Basanta Val, and Piotr Skowron. Proportional justified representation. *Artificial Intelligence*, page 104503, 2026.

Allen Schick. *The federal budget: Politics, policy, process*. Brookings Institution Press, 2008.

Yves Sintomer, Carsten Herzberg, and Anja Röcke. Participatory budgeting in europe: Potentials and challenges. *International Journal of Urban and Regional Research*, 32(1):164–178, 2008.

Piotr Skowron. A general theory of proportionality with additive utilities. *arXiv preprint arXiv:2602.08504*, 2026.

Nicholas Teh. The price of proportional representation in temporal voting. In *35th International Joint Conference on Artificial Intelligence, IJCAI 2026*, 2026.

Aaron B. Wildavsky. *The politics of the budgetary process*. Little, Brown, 1964. URL `https://cir.nii.ac.jp/crid/1971993809714340149`.

Valentin Zech, Niclas Boehmer, Edith Elkind, and Nicholas Teh. Multiwinner temporal voting with aversion to change. In *ECAI 2024*, pages 3236–3243. IOS Press, 2024.